

QUANTITATIVE VOLUME SPACE FORM RIGIDITY UNDER LOWER RICCI CURVATURE BOUND

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ABSTRACT. Let M be a compact n -manifold of $\text{Ric}_M \geq (n-1)H$ (H is a constant). We are concerned with the following space form rigidity: M is isometric to a space form of constant curvature H under either of the following conditions:

(i) There is $\rho > 0$ such that for any $x \in M$, the open ρ -ball at x^* in the (local) Riemannian universal covering space, $(U_\rho^*, x^*) \rightarrow (B_\rho(x), x)$, has the maximal volume i.e., the volume of a ρ -ball in the simply connected n -space form of curvature H .

(ii) For $H = -1$, the volume entropy of M is maximal i.e. $n-1$ ([LW1]).

The main results of this paper are quantitative space form rigidity i.e., statements that M is diffeomorphic and close in the Gromov-Hausdorff topology to a space form of constant curvature H , if M almost satisfies, under some additional condition, the above maximal volume condition. For $H = 1$, the quantitative spherical space form rigidity improves and generalizes the diffeomorphic sphere theorem in [CC2].

0. INTRODUCTION

Let M be a compact n -manifold of $\text{Ric}_M \geq (n-1)H$, H is a constant. The goal of this paper is to establish quantitative version for two space form rigidity under lower Ricci curvature bound (see Theorem 0.1 and 0.3). This work is based on, among other things, the work of Cheeger-Colding ([Ch], [Co1,2], [CC1,2]).

The first one is essentially the rigidity part of Bishop volume comparison. For our purpose (see Conjecture 0.15), we formulate it as follows. For a metric ball $B_r(x)$ on a manifold M , we will call $B_r(x^*)$ the rewinding of $B_r(x)$ and the volume, $\text{vol}(B_r(x^*))$, the rewinding volume of $B_r(x)$, where $\pi^* : (U_\rho^*, x^*) \rightarrow (B_\rho(x), x)$ is the (incomplete) Riemannian universal covering space.

Theorem 0.1. *Let M be a compact n -manifold of $\text{Ric}_M \geq (n-1)H$. If there is $\rho > 0$ such that for any $x \in M$, the rewinding volume $\text{vol}(B_\rho(x^*)) = \text{vol}(\underline{B}_\rho^H)$, then M is isometric to a space form of curvature H , where \underline{B}_ρ^H denotes a ρ -ball in the simply connected n -space form of constant curvature H .*

For $H \geq 0$, M in Theorem 0.1 may have an arbitrarily small volume i.e., collapsed. For $H = 1$, Theorem 0.1 includes the maximal volume rigidity: if a complete

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n -manifold M of $\text{Ric}_M \geq n - 1$ achieves the maximal volume (when $\rho = \pi$) i.e., the volume of unit sphere, then M is isometric to S_1^n .

A quantitative maximal volume rigidity is the following sphere theorem:

Theorem 0.2 ([CC2]). *There exists a constant $\epsilon(n) > 0$ such that for any $0 < \epsilon < \epsilon(n)$, if a compact n -manifold M satisfies*

$$\text{Ric}_M \geq n - 1, \quad \frac{\text{vol}(M)}{\text{vol}(S_1^n)} \geq 1 - \epsilon,$$

then M is diffeomorphic to the unit sphere, S_1^n , by a $\Psi(\epsilon|n)$ -isometry (i.e., a diffeomorphism with a distance distortion at most $\Psi(\epsilon|n)$), where $\Psi(\epsilon|n) \rightarrow 0$ as $\epsilon \rightarrow 0$ while n is fixed.

A homeomorphism in Theorem 0.2 was first obtained in [Pe1], a $\Psi(\epsilon|n)$ -closeness was established in [Co1], and Theorem 0.2 was proved in [CC2] via the Reifenberg's method.

The other space form rigidity result is the Ledrappier-Wang's maximal volume entropy rigidity ([LW1]). The volume entropy of a compact manifold M is defined by

$$h(M) = \lim_{R \rightarrow \infty} \frac{\ln(\text{vol}(B_R(\tilde{p})))}{R}, \quad \tilde{p} \in \tilde{M}$$

(for the existence of the limit, see [Ma]), where \tilde{M} denotes the Riemannian universal covering space of M . By Bishop volume comparison, for any compact n -manifold M of $\text{Ric}_M \geq -(n - 1)$, $h(M) \leq n - 1$, which equals to the volume entropy of any hyperbolic n -manifold.

Theorem 0.3 ([LW1]). *If a compact n -manifold M of $\text{Ric}_M \geq -(n - 1)$ achieves the maximal volume entropy i.e., $h(M) = n - 1$, then M is isometric to a hyperbolic manifold.*

We now begin to state our quantitative version for Theorem 0.1 with respect to rewinding volume and normalized $H = \pm 1$ and 0 respectively, starting with $H = 1$.

Theorem A. ⁴ *Given $n, \rho, v > 0$, there exists a constant $\epsilon(n, \rho, v) > 0$ such that for any $0 < \epsilon < \epsilon(n, \rho, v)$, if a compact n -manifold M satisfies*

$$\text{Ric}_M \geq n - 1, \quad \text{vol}(\tilde{M}) \geq v, \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^1)} \geq 1 - \epsilon, \quad \forall x \in M,$$

then M is diffeomorphic to a spherical space form by a $\Psi(\epsilon|n, \rho, v)$ -isometry, where $\text{vol}(B_\rho(x^))$ denotes the rewinding volume of $B_\rho(x)$.*

Theorem A generalizes and improves Theorem 0.2, see Remark 0.7. For $H = -1$, we have

Theorem B. *Given $n, \rho, d, v > 0$, there exists $\epsilon(n, \rho, d, v) > 0$ such that for any $0 < \epsilon < \epsilon(n, \rho, v, d)$, if a compact n -manifold M ($\tilde{p} \in \tilde{M}$) satisfies*

$$\text{Ric}_M \geq -(n - 1), \quad \text{diam}(M) \leq d, \quad \text{vol}(B_1(\tilde{p})) \geq v, \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^{-1})} \geq 1 - \epsilon, \quad \forall x \in M,$$

⁴In the early version arXiv:1604.06986, the non-collapsing condition was on M : “ $\text{vol}(M) \geq v > 0$ ”.

then M is diffeomorphic to a hyperbolic manifold by a $\Psi(\epsilon|n, \rho, d, v)$ -isometry.

Note that Theorem B does not hold if one removes a bound on diameter; there is a sequence of compact n -manifolds M_i ($n \geq 4$) of negative pinched sectional curvature $-1 \leq \sec_{M_i} \leq -1 + \epsilon_i$ and $\epsilon_i \rightarrow 0$ ($\text{diam}(M_i) \rightarrow \infty$), but M_i admits no hyperbolic metric ([GT]). On the other hand, given any $\rho, \epsilon > 0$, it is clear that for i large, $\frac{\text{vol}(B_\rho(\tilde{x}_i))}{\text{vol}(\underline{B}_\rho^{-1})} \geq 1 - \epsilon$ for any $\tilde{x}_i \in \tilde{M}_i$.

For $H = 0$, because of the splitting theorem of Cheeger-Gromoll ([CG]) we actually prove a rigidity result.

Theorem C. *Given $n, \rho, v > 0$, there exists $\epsilon = \epsilon(n, \rho, v) > 0$ such that if a compact n -manifold M satisfies*

$$\text{Ric}_M \geq 0, \quad \text{diam}(M) = 1, \quad \text{vol}(B_1(\tilde{p})) \geq v, \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^0)} \geq 1 - \epsilon, \quad \forall x \in M,$$

then M is isometric to a flat manifold.

A quantitative version of Theorem C is the following.

Theorem 0.4. *Given $n, \rho, v > 0$, there exist $\delta(n, \rho, v), \epsilon(n, \rho, v) > 0$ such that for any $0 < \delta < \delta(n, \rho, v)$, if a compact n -manifold M satisfies*

$$\text{Ric}_M \geq -(n-1)\delta, \quad 1 \geq \text{diam}(M), \quad \text{vol}(M) \geq v, \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^0)} \geq 1 - \epsilon(n, \rho, v), \quad \forall x \in M,$$

then M is diffeomorphic to a flat manifold by a $\Psi(\delta|n, \rho, v)$ -isometry.

Note that Theorem 0.4 does not hold if one relaxes the condition, ‘ $\text{vol}(M) \geq v$ ’, to ‘ $\text{vol}(B_1(\tilde{p})) \geq v$ ’. For instance, there is a sequence of compact nilpotent n -manifolds, N/Γ_i , which supports no flat metric, satisfying $|\sec_{N/\Gamma_i}| \leq \epsilon_i \rightarrow 0$, $\text{diam}(N/\Gamma_i) = 1$ and for all $\tilde{x}_i \in N$, $\frac{\text{vol}(B_1(\tilde{x}_i))}{\text{vol}(\underline{B}_1^0)} \rightarrow 1$ uniformly (cf. [Gr]).

We now state our quantitative version for Theorem 0.3.

Theorem D. *Given $n, d > 0$, there exists $\epsilon(n, d) > 0$ such that for any $0 < \epsilon < \epsilon(n, d)$, if a compact n -manifold M satisfies*

$$\text{Ric}_M \geq -(n-1), \quad d \geq \text{diam}(M), \quad h(M) \geq n-1-\epsilon,$$

then M is diffeomorphic to a hyperbolic manifold by a $\Psi(\epsilon|n, d)$ -isometry.

As discussed following Theorem B, Theorem D does not hold if one removes a bound on diameter.

To explore relations between Theorem B and Theorem D, we need the following property:

Theorem 0.5. *Let M_i be a sequence of compact n -manifold of $\text{Ric}_{M_i} \geq -(n-1)$ such that $M_i \xrightarrow{GH} M$. If M is a compact Riemannian n -manifold, then $h(M_i) \rightarrow h(M)$ as $i \rightarrow \infty$.*

Combining Theorem B, Theorem D and Theorem 0.5, we obtain the following corollary:

Corollary 0.6. *Let M be a compact n -manifold such that*

$$\text{Ric}_M \geq -(n-1), \quad \text{diam}(M) \leq d.$$

Then the following conditions are equivalent as $\epsilon \rightarrow 0$:

(0.6.1) M is diffeomorphic and ϵ -close to a hyperbolic manifold.

(0.6.2) $\frac{\text{vol}(B_1(\tilde{x}))}{\text{vol}(\underline{B}_1^{-1})} \geq 1 - \epsilon$, for any $\tilde{x} \in \tilde{M}$.

(0.6.3) $h(M) \geq n - 1 - \epsilon$.

A few remarks are in order:

Remark 0.7. Theorem A generalizes Theorem 0.2; first, if M has an almost maximal volume, then M is simply connected and thus M satisfies the conditions of Theorem A for $\rho = \pi + \sigma$ ($\sigma \ll 1$). Secondly, Theorem A applies to all spherical n -space form; all but finitely many are collapsed when n is odd. Theorem A also improves Theorem 0.2; if M in Theorem A is simply connected, then M is diffeomorphic and $\Psi(\epsilon|n)$ -close to S_1^n , while the conditions do not apriorily imply that the volume of M almost equals to $\text{vol}(S_1^n)$. We point it out that the case in Theorem A for $\rho = \pi + \sigma$ also recovers Theorem 4 in [Au] which is a generalization of Theorem 0.2.

Remark 0.8. If M satisfies the condition in Theorem B or Theorem D, then $\text{vol}(M)$ is not less than the volume of the hyperbolic metric on M ([BCG]), which is bounded below by a constant $v(n)$ (Heintze-Margulis, cf. [He]). In particular, this answers a question in [LW2] whether M of almost maximal volume entropy can collapse.

Remark 0.9. The gap phenomena in Theorem C that “ $\frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^0)} \geq 1 - \epsilon$ ” implies that “ $\frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^0)} = 1$ ” is related to the bounded ratio of diameters on M and \tilde{M} when $\pi_1(M)$ is finite ([KW]). Nevertheless, this volume gap phenomena seems not to be explored before; compare with flat manifolds rigidity under non-negative Ricci condition (e.g., Corollary 27 and 29, [Pet]).

Remark 0.10. Note that in Theorem 0.4, $\Psi(\delta|n, \rho, v)$ is independent of ϵ ; this is because a limit space of a sequence manifolds in Theorem 0.4 with $\delta_i \rightarrow 0$ is isometric to a flat manifold (see Lemma 4.7). The independence of ϵ was pointed out to us by S. Honda after the first version was put on ArXiv.

Remark 0.11. Let M be a compact hyperbolic n -manifold. The minimal volume rigidity in [BCG] says that any metric g on M of $\text{Ric}_g \geq -(n-1)$ satisfies that $\text{vol}(M, g) \geq \text{vol}(M)$, and “=” if and only if g is the hyperbolic metric on M . By Theorem 0.3, $h(M, g) \leq h(M)$ and “=” if and only if g coincides with the hyperbolic metric. In comparing the quantitative minimal volume rigidity (Theorem 1.3 in [BBCG]) with Theorem D, a substantial difference is that the former requires a non-collapsing condition but no condition on diameter, while the latter requires a bound on diameter but no non-collapsing condition.

Remark 0.12. For a special case of Theorem D that manifolds have strictly negative sectional curvature, see [LW2].

Remark 0.13. If, in Theorem A, B and D, the curvature condition is replaced by $\text{Ric}_M \geq (n-1)H$ ($H > 0$ or $H < 0$), then conclusions hold with respect to the space form of constant curvature H , provided that ϵ also depends on H .

Remark 0.14. The volume conditions in Theorem A-C imply that the Riemannian universal covering space satisfies that for any $\tilde{x} \in \tilde{M}$, $\frac{\text{vol}(B_{\rho'}(\tilde{x}))}{\text{vol}(\underline{B}_{\rho'}^H)} \geq 1 - \Psi(\epsilon|n, \rho, d, v)$ ($H = 1, -1$, or 0), where $\rho' = \rho'(n, \rho, d, v) > 0$, see Corollary 3.3.

In the light of Theorem A-C, we propose the following:

Conjecture 0.15. (Quantitative volume space form rigidity) Given $n, \rho > 0$ and $H = \pm 1$ or 0 , there exists a constant $\epsilon(n, \rho) > 0$ such that for any $0 < \epsilon < \epsilon(n, \rho)$, if a compact n -manifold M satisfies

$$\text{Ric}_M \geq (n-1)H, \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon, \quad \forall x \in M,$$

then M is diffeomorphic and $\Psi(\epsilon|n, \rho)$ -close to a space form of constant curvature H , provided that $\text{diam}(M) \leq d$ (and thus $\epsilon(n, \rho, d)$) when $H \neq 1$.

The following is a supporting evidence for Conjecture 0.15 (see [CRX]).

Theorem E. *Conjecture 0.15 holds for the class of Einstein manifolds.*

We now briefly describe our approach to Theorem A-C and Theorem D which is quite involved with tools from several fields. The most significant tool is from the Cheeger-Colding theory ([Ch], [Co2], [CC1-3]) and the Perel'man's pseudo-locality of Ricci flows ([BW], [Ha1,2], [Pe2]). In our proof of Theorem A, we established a C^0 -convergence (see Theorem 2.7), and in the proof of Theorem D, we establish that an almost volume annulus of fixed width and radius going to ∞ ($H \leq 0$) contains a large ball that is almost metric warped product (see Theorem 1.4). This result complements the Cheeger-Colding's theorem that an almost volume annulus (of bounded radius) is an almost metric annulus, and also yields a new proof of Theorem 0.3 (see Remark 4.5) that does not rely on [LiW] (cf. [LW1], [Li]).

Starting with a contradicting sequence to Theorem A-C, $M_i \xrightarrow{GH} X$, such that $\frac{\text{vol}(B_\rho(x_i^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon_i$ for all $x_i \in M_i$, and we will study the associate equivariant sequence of the Riemannian universal covering spaces, which satisfies the following commutative diagram ([FY]):

$$(0.16) \quad \begin{array}{ccc} (\tilde{M}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{GH} & (\tilde{X}, \tilde{p}, G) \\ \downarrow \pi_i & & \downarrow \pi \\ (M_i, p_i) & \xrightarrow{GH} & (X, p), \end{array}$$

where $\Gamma_i = \pi_1(M_i, p_i)$ is the fundamental group, G is the limiting Lie group ([CC3]) and the identity component G_0 is nilpotent ([KW]). We will first show that \tilde{X} is locally isometric to a space form. For any $\tilde{x} \in \tilde{X}$, let $\tilde{x}_i \in \tilde{M}_i$ such that $\tilde{x}_i \rightarrow \tilde{x}$, we study a local version of (0.16):

$$(0.17) \quad \begin{array}{ccc} (U_\rho^*, x_i^*, \Lambda_i) & \xrightarrow{GH} & (\tilde{Y}, x^*, K) \\ \downarrow \tilde{\pi}_i^* & & \downarrow \tilde{\pi}^* \\ (\pi_i^{-1}(B_\rho(x_i)), \tilde{x}_i) & \xrightarrow{GH} & (Y, \tilde{x}), \end{array}$$

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where $\Lambda_i = \pi_1(\pi_i^{-1}(B_\rho(x_i)), \tilde{x}_i)$. According to the Cheeger-Colding's theorem that an almost volume annulus is an almost metric annulus, $\frac{\text{vol}(B_\rho(x_i^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon_i$ implies that $d_{GH}(B_{\frac{\rho}{2}}(x_i^*), \underline{B}_{\frac{\rho}{2}}^H) < \Psi(\epsilon_i | n, \rho, d, v)$ (Theorem 1.2), and thus \tilde{Y} is locally isometric to a H -space form. Since \tilde{M}_i is not collapsed, K is discrete. It remains to check that K acts freely (Theorem 2.1), thus a small ball at \tilde{x} is isometric to a small ball in the n -space form. If $e \neq \gamma \in K$ and $q^* \in B_{\frac{\rho}{4}}(x^*)$ such that $\gamma(q^*) = q^*$, under the non-collapsing equivariant convergence we show that γ and q^* can be chosen so that there are $\gamma_i \in \Lambda_i$ of order equal to that of γ , $\gamma_i \rightarrow \gamma$, $q_i^* \rightarrow q^*$ and the displacement of γ_i at q_i^* , $\mu_i \rightarrow 0$, is almost minimum around q_i^* . In our circumstance, the rescaling sequence, $(\mu_i^{-1}U_\rho^*, q_i^*, \langle \gamma_i \rangle) \xrightarrow{GH} (\mathbb{R}^n, \tilde{q}, \langle \gamma' \rangle)$, which leads to a contradiction because γ' must fix some point in \mathbb{R}^n , while γ_i moves every point at least a definite amount, where $\langle \gamma_i \rangle$ denotes the subgroup generated by γ_i .

If G is discrete, similar to the above we conclude that G acts freely on \tilde{X} (Theorem 2.1), and thus X is isometric to an n -space form. We then get a contradiction by applying the diffeomorphic stability theorem in [CC2]. For $H = -1$, we will show that G is discrete (Theorem 2.5): using the nilpotency of G_0 and the compactness of \tilde{X}/G we show that G_0 contains neither elliptic nor hyperbolic elements (Lemma 2.6). Using (0.16), we construct a geodesic segment in some G_0 -orbit, and thus conclude that G_0 contains no parabolic element i.e., $G_0 = e$. This finishes the proof of Theorem B.

For $H = 0$, $\tilde{X} = \mathbb{R}^k \times F$ and $\tilde{M}_i = \mathbb{R}^k \times N_i$ (Cheeger-Gromoll splitting theorem), where F is a compact flat manifold, and N_i is a compact simply connected manifold of non-negative Ricci curvature. We show that $\text{diam}(N_i)$ is uniformly bounded above, and thus applying the diffeomorphic stability theorem in [CC2] we derive a contradiction.

For $H = 1$, in (0.16) we may assume an ϵ_i -equivariant diffeomorphism, $\tilde{h}_i : (\tilde{M}_i, \Gamma_i) \rightarrow (S_1^n, G)$ ([CC2]). Via \tilde{h}_i , we identify (M_i, Γ_i) as a free Γ_i -action on S_1^n by ϵ_i -isometries. By [MRW], for i large there is an injective homomorphism, $\phi_i : \Gamma_i \rightarrow G$ (see Lemma 3.4). We show that the $\phi_i(\Gamma_i)$ -action on S_1^n is free (see (3.5.1)). By now we can perform the center of mass to perturb $\text{id}_{S_1^n}$ to a map, $\tilde{f}_i : S^n \rightarrow S^n$, that commutes the Γ_i -action with the $\phi_i(\Gamma_i)$ -action. It remains to show that \tilde{f}_i is a diffeomorphism, and thus a contradiction. According to [GK], \tilde{f}_i is a diffeomorphism when the Γ_i - and $\phi_i(\Gamma_i)$ -actions are close in C^1 -norm. To see it, we will use Ricci flows of \tilde{g}_i : using Perelman's pseudo-locality ([Pe2]) and a distance estimate in [BW] we show that a solution $\tilde{g}_i(t)$ is C^0 -close to g_1^1 on S_1^n (see Theorem 2.7); which is also locally $C^{1,\alpha}$ -close to g_1^1 up to a definite rescaling. Since Γ_i remains to be isometries with respect to $\tilde{g}_i(t)$, the above regularities guarantee the desired C^1 -closeness (see (3.5.2)).

In the proof of Theorem D, we again start with a contradicting sequence as in (0.16), and it suffices to show that \tilde{X} is isometric to \mathbb{H}^n , and by the volume convergence ([Co2]) M_i satisfies the conditions of Theorem B, a contradiction. Fixing $R > 50d$, we will prove that $d_{GH}(B_R(\tilde{p}_i), \underline{B}_R^{-1}) < \Psi(\epsilon_i | n, d, R)$, where \underline{B}_R^{-1} is a ball in \mathbb{H}^k for some $k \leq n$ (Lemma 4.4). First, following [Li] we show that $h(M) \geq n - 1 - \epsilon$ implies a sequence, $r_i \rightarrow \infty$, such that the ratio, $\lim_{i \rightarrow \infty} \frac{\text{vol}(\partial B_{r_i+50R}(\tilde{p}))}{\text{vol}(\partial B_{r_i-50R}(\tilde{p}))} \geq e^{100R(n-1-\epsilon)}$, which approximates the limit of the same type ratio on \mathbb{H}^n . Because $\text{vol}(A_{r_i-50R, r_i+50R}(\tilde{p})) \rightarrow \infty$ as $r_i \rightarrow \infty$, the

Cheeger-Colding's theorem that an almost volume annulus is an almost metric annulus cannot be applied in our situation. Instead, we establish the following (weak) property (see Theorem 1.4): annulus $A_{r_i-50R, r_i+50R}(\tilde{p})$ contains a ball $B_{2R}(\tilde{q}_i)$ such that $d_{GH}(B_{2R}(\tilde{q}_i), \underline{B}_{2R}^{-1}) < \Psi(\epsilon_i, r_i^{-1}|n, d, R)$, which leads to the desired estimate via pullback $B_{2R}(\tilde{q}_i)$ to $B_{2R}(\gamma_i(\tilde{q}_i)) \supseteq B_R(\tilde{p})$ with suitable element $\gamma_i \in \Gamma_i$.

The remaining proof is to show that $k = n$. We will show that $\dim(X) = n$. If $k = \dim(X) < n$, then M_i is collapsed. By [FY] (see Lemma 1.13), there is $\epsilon > 0$ such that the subgroup $\Gamma_i^\epsilon \subset \Gamma_i$ generated by elements whose displacement on $B_1(\tilde{p}_i)$ are uniformly smaller than ϵ converges to G_0 . From the proof of Theorem B, G_0 is trivial and thus Γ_i^ϵ is finite. Since $h(M_i)$ can be calculated in terms of the growth of $\pi_1(M_i)$ at \tilde{p}_i , via center of mass method we construct a $\Gamma_i/\Gamma_i^\epsilon$ -conjugate map from $(\tilde{M}_i/\Gamma_i^\epsilon, \Gamma_i/\Gamma_i^\epsilon) \rightarrow (\mathbb{H}^k, G)$ which is also an ϵ_i -Gromov-Hausdorff approximation when restricting to $B_R(\tilde{p}_i)$ (Lemma 4.7), we are able to estimate $h(M_i) \leq k - 1 + \epsilon_i$ (Theorem 4.6), a contradiction.

The rest of the paper is organized as follows:

In Section 1, we supply basic notions and tools concerning a convergent sequence of compact n -manifolds with Ricci curvature bounded below and diameter bounded above, which will be freely used through the rest of the paper. In particular, we will state our result that an asymptotic volume annulus contains many disjoint balls of almost warped product structure (see Theorem 1.4), which provides information complements to the Cheeger-Colding's theorem that almost volume annulus is almost metric annulus (Theorem 1.3).

In section 2, we will establish three key properties for our proofs of Theorems A-C and D: a sufficient condition for a limiting group G to act freely on a limit space \tilde{X} (Theorem 2.1), for $H = -1$, G is discrete (Theorem 2.5) and a C^0 -convergence of Ricci flows associate to a sequence of GH-convergence with Ricci curvature bounded below (Theorem 2.7).

In Section 3, we will prove Theorem A-C, Theorem E and Theorem 0.4.

In Section 4, we will prove Theorem D by assuming Theorem 1.4. We will also prove Theorem 0.5 and Corollary 0.6.

In Section 5, we will prove Theorem 1.4.

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1. PRELIMINARIES

The purpose of this section is to supply notions and basic properties from the fundamental work of Cheeger-Colding on degeneration of Riemannian metrics with Ricci curvature bounded from below, as well as those related to equivariant Gromov-Hausdorff convergence. These will be used through out this paper, and we refer the readers to [Ch], [CC1-3], [Co1,2] and [FY] for details.

We will also state our result that an almost volume annulus of fixed width and large radius contains many disjoint balls with almost warped product structure (see Theorem 1.4).

a. Manifolds of Ricci curvature bounded below.

Let N be a Riemannian $(n-1)$ -manifold, let $k : (a, b) \rightarrow \mathbb{R}$ be a smooth positive function and let $(a, b) \times_k N$ be the k -warped product whose Riemannian tensor is

$$g = dr^2 + k^2(r)g_N.$$

The Riemannian distance $|(r_1, x_1)(r_2, x_2)|$ ($x_1 \neq x_2$) equals to the infimum of the length

$$\int_0^l \sqrt{(c_1'(t))^2 + k^2(c_1(t))} dt$$

for any smooth curve $c(t) = (c_1(t), c_2(t))$ such that $c(0) = (r_1, x_1)$, $c(l) = (r_2, x_2)$ and $|c_2'| \equiv 1$, and $|(r_1, x)(r_2, x)| = |r_2 - r_1|$. Thus given a, b, k , there is a function (e.g., the law of cosine on space forms)

$$\rho_{a,b,k}(r_1, r_2, |x_1 x_2|) = |(r_1, x_2)(r_2, x_2)|.$$

Using the same formula for $|(r_1, x_1)(r_2, x_2)|$, one can extend the k -warped product $(a, b) \times_k Y$ to any metric space Y (not necessarily a length space); see [CC1].

We first recall the following Cheeger-Colding's "almost volume warped product implies almost metric warped product" theorem.

Theorem 1.1 ([CC1]). *Let M be a Riemannian manifold, let r be a distance function to a compact subset in M , let $0 < \alpha' < \alpha$, $\alpha - \alpha' > 2\xi > 0$, let $A_{a,b} = r^{-1}((a, b))$ and let*

$$\mathcal{V}(\xi) = \inf \left\{ \frac{\text{vol}(B_\xi(q))}{\text{vol}(A_{a,b})} \mid \text{for all } q \in A_{a,b} \text{ with } B_\xi(q) \subset A_{a,b} \right\}.$$

If

$$\text{Ric}_M \geq -(n-1) \frac{k''(a)}{k(a)} \quad (\text{on } r^{-1}(a)),$$

$$\Delta r \leq (n-1) \frac{k'(a)}{k(a)} \quad (\text{on } r^{-1}(a)),$$

$$(1.1.1) \quad \frac{\text{vol}(A_{a,b})}{\text{vol}(r^{-1}(a))} \geq (1 - \epsilon) \frac{\int_a^b k^{n-1}(r) dr}{k^{n-1}(a)}.$$

Then there exists a length metric space Y , with at most $\#(a, b, k, \mathcal{V}(\xi))$ components Y_i , satisfying

$$\text{diam}(Y_i) \leq D(a, b, k, \mathcal{V}(\xi)),$$

such that

$$(1.1.2) \quad d_{GH}(A_{a+\alpha, b-\alpha}, (a+\alpha, b-\alpha) \times_k Y) \leq \Psi(\epsilon | n, k, a, b, \alpha', \xi, \mathcal{V}(\xi))$$

with respect to the two metrics $d^{\alpha', \alpha}$ and $\underline{d}^{\alpha', \alpha}$, where $d^{\alpha', \alpha}$ (resp. $\underline{d}^{\alpha', \alpha}$) denotes the restriction of the intrinsic metric of $A_{a+\alpha', b-\alpha'}$ on $A_{a+\alpha, b-\alpha}$ (resp. $(a+\alpha', b-\alpha') \times_k Y$) on $(a+\alpha, b-\alpha) \times_k Y$.

Let

$$\text{sn}_H(r) = \begin{cases} \frac{\sin \sqrt{H}r}{\sqrt{H}} & H > 0 \\ r & H = 0 \\ \frac{\sinh \sqrt{-H}r}{\sqrt{-H}} & H < 0 \end{cases}.$$

Applying Theorem 1.1 to $\text{sn}_H(r)$ with $r(x) = d(p, x) : M \rightarrow \mathbb{R}$, we conclude the following "almost maximal volume ball implies almost space form ball", which is important to our work (one may need to shift the center a bit to see the following).

Theorem 1.2. For $n, \rho, \epsilon > 0$, if a complete n -manifold M contains a point p satisfies

$$\text{Ric}_M \geq (n-1)H, \quad \frac{\text{vol}(B_\rho(p))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon,$$

then $d_{GH}(B_{\frac{\rho}{2}}(p), \underline{B}_{\frac{\rho}{2}}^H) < \Psi(\epsilon|n, \rho, H)$.

Another important application of Theorem 1.1 is the following an “almost volume annulus” is an “almost metric annulus”. For $p \in M$, $L > 2R > 0$, let $A_{L-2R, L+2R}(p) = \{x \in M, L-2R < |xp| < L+2R\}$.

Theorem 1.3. Given $n, H \leq 0, L > 2R > 0$, if a complete n -manifold M contains a point p satisfies

$$(1.3.1) \quad \text{Ric}_M \geq (n-1)H, \quad \frac{\text{vol}(\partial B_{L-2R}(p))}{\text{vol}(\partial \underline{B}_{L-2R}^H)} \leq (1 + \epsilon) \frac{\text{vol}(A_{L-2R, L+2R}(p))}{\text{vol}(\underline{A}_{L-2R, L+2R}^H)},$$

then

$$(1.3.2) \quad d_{GH}(A_{L-R, L+R}(p), (L-R, L+R) \times_{\text{sn}_H(r)} Y) \leq \Psi(\epsilon|n, L, R, H),$$

where Y is a length metric space (may be not connected).

It turns out that in our proof of Theorem D, the condition that $h(M) \geq n-1-\epsilon$ implies that (1.3.1) is satisfied asymptotically i.e., only as $L \rightarrow \infty$ (see Lemma 4.2). Because in our circumstance $\text{vol}(A_{L-R, L+R}(p)) \rightarrow \infty$ as $L \rightarrow \infty$, it is not possible to have (1.3.2) in our circumstance.

In our proof Theorem D, it is crucial for us to establish the following result.

Theorem 1.4. Given $n, H \leq 0, L \gg R \geq \rho > 0, \epsilon > 0$, there exists a constant $c = c(n, H, R, \rho)$ such that if a complete n -manifold M contains a point p satisfies (1.3.1), then there are disjoint ρ -balls, $B_\rho(q_i) \subset A_{L-R, L+R}(p)$, for each $B_\rho(q_i)$,

$$(1.4.1) \quad d_{GH}(B_\rho(q_i), B_\rho((0, x_i))) \leq \Psi(\epsilon, L^{-1}|n, H, R, \rho)$$

where $B_\rho((0, x_i)) \subset \mathbb{R}^1 \times_{e\sqrt{-H}r} Y_i$ for some length metric space Y_i , and

$$(1.4.2) \quad \frac{\text{vol}(\bigcup_i B_\rho(q_i))}{\text{vol}(A_{L-R, L+R}(p))} \geq c(n, H, R, \rho).$$

In particular, for $H = 0$, we have that each $B_\rho(q_i)$ is almost splitting.

Roughly, Theorem 1.4 says that for any fixed $R > 0$, if $A_{L-2R, L+2R}(p)$ is an almost volume annulus as $L \rightarrow \infty$, then (even if its volume blows up to infinity) one can have lots of disjoint balls of fixed radius $\rho \leq R$ in the annulus, each of which is close to a ball in a metric annulus.

The proof of Theorem 1.4 uses the same techniques from [Ch] and [CC1], and because it is technical and tedious, we will leave the proof in section 5.

Remark 1.5. The almost volume annulus condition (1.3.1) implies the following:

$$(1.5.1) \quad \frac{\text{vol}(\partial B_{L+R}(p))}{\text{vol}(\partial \underline{B}_{L+R}^H)} \geq (1 - \Psi(\epsilon|n, H, R)) \frac{\text{vol}(\partial B_{L-R}(p))}{\text{vol}(\partial \underline{B}_{L-R}^H)}$$

From the proof of Theorem 1.3 in [CC1], one sees that indeed only (1.5.1) is applied. Furthermore, (1.3.1) and (1.5.1) are equivalent conditions when ϵ is small.

Consider a sequence of complete n -manifolds, $(M_i, p_i) \xrightarrow{GH} (X, p)$, such that $\text{Ric}_{M_i} \geq -(n-1)$. If M_i is not collapsed, then a basic property is:

Theorem 1.6 ([Co2,CC2]). *Let $(M_i, p_i) \xrightarrow{GH} (X, p)$ such that $\text{Ric}_{M_i} \geq -(n-1)$. If $\text{vol}(B_1(p_i)) \geq v > 0$, then for any $r > 0$, $M_i \ni x_i \rightarrow x \in X$, $\text{vol}(B_r(x_i)) \rightarrow \text{Haus}^n(B_r(x))$, where Haus^n denotes the n -dimensional Hausdorff measure.*

Let X be a complete separable length metric space. A point $x \in X$ is called a (ϵ, r) -Reifenberg point, if for any $0 < s < r$,

$$d_{GH}(B_s(x), \underline{B}_s^0) \leq \epsilon s.$$

X is called a (ϵ, r) -Reifenberg space if every point in X is a (ϵ, r) -Reifenberg point.

Theorem 1.7 ([CC2]). *Let $M_i \xrightarrow{GH} X$ be a sequence of complete n -manifolds of $\text{Ric}_{M_i} \geq -(n-1)$. Then there is a constant $\epsilon(n) > 0$ such that for i large*

(1.7.1) *If X is a Reifenberg (ϵ, r) -space with $\epsilon < \epsilon(n)$, then there is a homeomorphic bi-Hölder equivalence between M_i and X .*

(1.7.2) *If X is a Riemannian manifold, then there is a diffeomorphic bi-Hölder equivalence between M_i and X .*

Theorem 1.8 ([CC3]). *Let $(M_i, p_i) \xrightarrow{GH} (X, p)$ such that $\text{Ric}_{M_i} \geq -(n-1)$. If $\text{vol}(B_1(p_i)) \geq v > 0$, then the isometry group of X is a Lie group.*

Theorem 1.8 holds for any limit space of Riemannian n -manifolds with Ricci curvature bounded below ([CN]).

According to the classical Margulis Lemma, if M is a symmetric space, the subgroup of the fundamental group of M generated by loops of small length is virtually nilpotent. Margulis Lemma was extended in [FY] to manifolds of $\text{sec} \geq -1$ that the subgroup is virtually nilpotent, and in [KPT] a bound on the index of the nilpotent subgroup was obtained depending only on n . Recently, Kapovitch-Wilking proved the following generalized Margulis Lemma (conjectured by Gromov):

Theorem 1.9 ([KW]). *There are constants $\epsilon(n), w(n) > 0$ if M is a complete n -manifold of $\text{Ric}_M \geq -(n-1)$, $p \in M$, then the image subgroup, $\text{Im}(\pi_1(B_\epsilon(p)) \rightarrow \pi_1(M))$ contains a nilpotent subgroup of index $\leq w(n)$.*

b. Equivariant Gromov-Hausdorff convergence.

The reference of this part is [FY] (cf. [Ro2]).

Let $X_i \xrightarrow{GH} X$ be a convergent sequence of compact length metric spaces, i.e., there are a sequence $\epsilon_i \rightarrow 0$ and a sequence of maps $h_i : X_i \rightarrow X$, such that $||h_i(x_i)h_i(x'_i)|_X - |x_i x'_i|_{X_i}| < \epsilon_i$ (ϵ_i -isometry), and for any $x \in X$, there is $x_i \in X_i$ such that $|h_i(x_i)x|_X < \epsilon_i$ (ϵ_i -onto), and h_i is called an ϵ_i -Gromov-Hausdorff approximation, briefly, ϵ_i -GHA. From now on, we will omit the subindex in the distance function “ $|\cdot|$ ”.

Assume that X_i admits a closed group Γ_i -action by isometries. Then $(X_i, \Gamma_i) \xrightarrow{GH} (X, \Gamma)$ means that there are a sequence $\epsilon_i \rightarrow 0$ and a sequence of (h_i, ϕ_i, ψ_i) , $h_i : X_i \rightarrow X$, $\phi_i : \Gamma_i \rightarrow \Gamma$ and $\psi_i : \Gamma \rightarrow \Gamma_i$ which are ϵ_i -GHAs such that for all $x_i \in X_i, \gamma_i \in \Gamma_i$ and $\gamma \in \Gamma$,

$$(1.10) \quad |h_i(x_i)[\phi_i(\gamma_i)h_i(\gamma_i^{-1}(x_i))]| < \epsilon_i, \quad |h_i(x_i)[\gamma^{-1}(h_i(\psi_i(\gamma)(x_i)))]| < \epsilon_i,$$

where Γ is a closed group of isometries on X , Γ_i and Γ are equipped with the induced metrics from X_i and X . We call (h_i, ϕ_i, ψ_i) an ϵ_i -equivariant GHA.

When X is not compact, then the above notion of equivariant convergence naturally extends to a pointed version (h_i, ϕ_i, ψ_i) : $h_i : B_{\epsilon_i^{-1}}(p_i) \rightarrow B_{\epsilon_i^{-1} + \epsilon_i}(p)$, $h_i(p_i) = p$, $\phi_i : \Gamma_i(\epsilon_i^{-1}) \rightarrow \Gamma(\epsilon_i^{-1} + \epsilon_i)$, $\phi_i(e_i) = e$, $\psi_i : \Gamma(\epsilon_i^{-1}) \rightarrow \Gamma_i(\epsilon_i^{-1} + \epsilon_i)$, $\psi_i(e) = e_i$, and (1.10) holds whenever the multiplications stay in the domain of h_i , where $\Gamma_i(R) = \{\gamma_i \in \Gamma_i, |p_i \gamma_i(p_i)| \leq R\}$.

Lemma 1.11. *Let $(X_i, p_i) \xrightarrow{GH} (X, p)$, where X_i is a complete locally compact length space. Assume that Γ_i is a closed group of isometries on X_i . Then there is a closed group G of isometries on X such that passing to a subsequence, $(X_i, p_i, \Gamma_i) \xrightarrow{GH} (X, p, G)$.*

Lemma 1.12. *Let $(X_i, p, \Gamma_i) \xrightarrow{GH} (X, p, G)$, where X_i is a complete locally compact length space and Γ_i is a closed subgroup of isometries. Then $(X_i/\Gamma_i, \bar{p}_i) \xrightarrow{GH} (X/G, \bar{p})$.*

For $p_i \in X_i$, let $\Gamma_i = \pi_1(X_i, p_i)$ be the fundamental group. Assume that the universal covering space, $\pi_i : (\tilde{X}_i, \tilde{p}_i) \rightarrow (X_i, p_i)$, exists.

Lemma 1.13. *Let $X_i \xrightarrow{GH} X$ be a sequence of compact length metric space. Then passing to a subsequence the following diagram commutes,*

$$\begin{array}{ccc} (\tilde{X}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{GH} & (\tilde{X}, \tilde{p}, G) \\ \downarrow \pi_i & & \downarrow \pi \\ (X_i, p_i) & \xrightarrow{GH} & (X, p). \end{array}$$

If X is compact and G/G_0 is discrete, then there is $\epsilon > 0$ such that the subgroup, Γ_i^ϵ , generated by elements with displacement bounded above by ϵ on $B_{2d}(\tilde{p}_i)$, is normal and for i large, $\Gamma_i/\Gamma_i^\epsilon \xrightarrow{\text{isom}} G/G_0$.

Combining Lemma 1.12 and 1.13, we obtain the following commutative diagram:

$$(1.14) \quad \begin{array}{ccc} (\tilde{X}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{GH} & (\tilde{X}, \tilde{p}, G) \\ \downarrow \hat{\pi}_i & & \downarrow \hat{\pi} \\ (\hat{X}_i, \hat{p}_i, \hat{\Gamma}_i) & \xrightarrow{GH} & (\hat{X}, \hat{p}, \hat{G}) \\ \downarrow \bar{\pi}_i & & \downarrow \bar{\pi} \\ (X_i, p_i) & \xrightarrow{GH} & (X, p), \end{array}$$

where $\hat{X}_i = \tilde{X}_i/\Gamma_i^\epsilon$, $\hat{X} = \tilde{X}/G_0$, $\hat{\Gamma}_i = \Gamma_i/\Gamma_i^\epsilon$ and $\hat{G} = G/G_0$.

2. THE FREE ACTION, THE DISCRETENESS OF LIMITING GROUPS AND THE C^0 -CONVERGENCE

In this section, we will establish three key properties for our proofs of Theorems A, B and D: Theorem 2.1, Theorem 2.5 and Theorem 2.7.

a. Free limit isometric actions.

Let (M_i, p_i) be a sequence of complete n -manifolds, let $\pi_i^* : (U_d^*, p_i^*) \rightarrow (B_d(p_i), p_i)$ be the Riemannian universal covering spaces, and let $\Lambda_i = \pi_1(B_d(p_i), p_i)$ denote the fundamental group.

Theorem 2.1. *Given $n, d, v, r > 0$, there exists a constant $\epsilon = \epsilon(n, v) > 0$ such that if a sequence of complete n -manifolds, (M_i, p_i) , satisfies*

$$\text{Ric}_{M_i} \geq -(n-1), \quad \text{vol}(B_1(p)) \geq v, \quad \forall x^* \in U_d^* \text{ is a } (\epsilon, r)\text{-Reifenberg point}$$

and the following commutative diagram:

$$\begin{array}{ccc} (U_d^*, p_i^*, \Lambda_i) & \xrightarrow{GH} & (X^*, p^*, K) \\ \downarrow \pi_i^* & & \downarrow \pi^* \\ (B_d(p_i), p_i) & \xrightarrow{GH} & (B_d(p), p), \end{array}$$

then the discrete group K acts freely on $B_{\frac{d}{4}}(p^)$ i.e., K has no isotropy group in $B_{\frac{d}{4}}(p^*)$.*

Corollary 2.2. *Given $n, \rho, v > 0$ and $H \geq -1$, there exists a constant $\epsilon = \epsilon(n, v) > 0$ such that if a sequence of complete n -manifolds, (M_i, p_i) , satisfies*

$$\text{Ric}_{M_i} \geq (n-1)H_i \rightarrow (n-1)H, \quad \text{vol}(B_1(p)) \geq v, \quad \frac{\text{vol}(B_\rho(p_i^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon,$$

and the following commutative diagram:

$$(2.2.1) \quad \begin{array}{ccc} (U_\rho^*, p_i^*, \Lambda_i) & \xrightarrow{GH} & (\tilde{X}, p^*, K) \\ \downarrow \pi_i^* & & \downarrow \pi^* \\ (B_\rho(p_i), p_i) & \xrightarrow{GH} & (B_\rho(p), p), \end{array}$$

where $\pi_i^ : (U_\rho^*, p_i^*) \rightarrow (B_\rho(p_i), p_i)$ is the Riemannian universal cover, and $\Lambda_i = \pi_1(B_\rho(p_i), p_i)$. Then K acts freely on $B_{\frac{\rho}{4}}(p^*)$ i.e., K has no isotropy group in $B_{\frac{\rho}{4}}(p^*)$.*

In the proof, we will use the following lemma due to [PR]:

Lemma 2.3. *Let $(M_i, p_i) \xrightarrow{GH} (X, p)$ be a sequence of complete n -manifolds satisfying*

$$\text{Ric}_{M_i} \geq (n-1)H_i \rightarrow (n-1)H, \quad \text{vol}(B_\rho(p_i^*)) \geq v > 0,$$

and the commutative diagram (2.2.1). If a subgroup H_i of Λ_i satisfies that $H_i \rightarrow e \in K$, then for i large, $H_i = e$.

Proof. Arguing by contradiction, without loss of generality we may assume $e \neq \gamma_i \in H_i$ for all i such that the following diagram commutes:

$$\begin{array}{ccc} (U_\rho^*, p_i^*, \langle \gamma_i \rangle) & \xrightarrow{GH} & (\tilde{X}, p^*, e) \\ \downarrow \hat{\pi}_i^* & & \downarrow \hat{\pi}^* \\ (U_\rho^* / \langle \gamma_i \rangle, \hat{p}_i) & \xrightarrow{GH} & (\hat{X}, \hat{p}), \end{array}$$

where $\langle \gamma_i \rangle$ denotes the subgroup generated by $\gamma_i \in \Lambda_i$. Since $\langle \gamma_i \rangle \xrightarrow{GH} e$, by Lemma 1.12 $\tilde{X} = \hat{X}$, $B_r(p_i^*)$ and $\gamma_i(B_r(p_i^*)) \subset B_{r+\epsilon_i}(p_i^*)$ for some $\epsilon_i \rightarrow 0$. Let D_i denote a (Dirichlet) fundamental domain of $U_\rho^*(p_i)/\langle \gamma_i \rangle$ at p_i^* . Then for $0 < r < \frac{\rho}{2}$, $[B_r(p_i^*) \cap D_i] \cap [\gamma_i(B_r(p_i^*) \cap D_i)] = \emptyset$. Since $\text{vol}(B_\rho(p_i^*)) \geq v > 0$, we are able to apply Theorem 1.6 to derive

$$\begin{aligned} \text{Haus}^n(B_r(p^*)) &= \text{Haus}^n(B_r(\hat{p})) = \lim_{i \rightarrow \infty} \text{vol}(B_r(\hat{p}_i)) = \lim_{i \rightarrow \infty} \text{vol}(B_r(p_i^*) \cap D_i) \\ &= \lim_{i \rightarrow \infty} \frac{1}{2} [\text{vol}(B_r(p_i^*) \cap D_i) + \text{vol}(\gamma_i(B_r(p_i^*) \cap D_i))] \\ &\leq \lim_{i \rightarrow \infty} \frac{1}{2} \text{vol}(B_{r+\epsilon_i}(p_i^*)) = \frac{1}{2} \text{Haus}^n(B_r(p^*)), \end{aligned}$$

a contradiction. \square

Proof of Theorem 2.1.

Arguing by contradiction, assume a sequence, $(\epsilon_j, r_j) \rightarrow (0, 0)$, and for each j , there is a contradicting sequence $(M_{i,j}, p_{i,j})$ to Theorem 2.1,

$$\begin{array}{ccc} (U_d^*, p_{i,j}^*, \Lambda_{i,j}) & \xrightarrow{GH} & (\tilde{Y}_j, p_j^*, K_j) \\ \downarrow \pi_{i,j} & & \downarrow \pi_j \\ (B_d(p_{i,j}), p_{i,j}) & \xrightarrow{GH} & (B_d(p_j), p_j), \end{array}$$

such that

$\text{Ric}_{M_{i,j}} \geq -(n-1)$, $\text{vol}(B_1(p_{i,j})) \geq v$, $\forall x_{i,j}^* \in B_1(p_{i,j}^*)$ is a (ϵ_j, r_j) -Reifenberg point,

and K_j has an isotropy group in $B_{\frac{d}{4}}(p_j^*)$. Passing to a subsequence, we may assume

$$(\tilde{Y}_j, p_j^*, K_j) \xrightarrow{GH} (\tilde{Y}, p^*, K).$$

Assume $e_j \neq \gamma_j \in K_j$, $q_j^* \in B_{\frac{d}{4}}(p_j^*)$ such that $\langle \gamma_j \rangle(q_j^*) = q_j^*$. Passing to a subsequence, we may assume $\langle \gamma_j \rangle \rightarrow W$ and $q_j^* \rightarrow q^*$ such that $W(q^*) = q^*$. We observe that Lemma 2.3 can still apply to the above sequence i.e., if $\gamma_j \in K_j$ such that $\langle \gamma_j \rangle \rightarrow e$, then $\gamma_j = e$ for j large. Hence $W \neq e$.

Without loss of generality, we may assume that $q_j^* = p_j^*$. For $e \neq \gamma \in W$, $\gamma(p^*) = p^*$. Since $\text{vol}(B_d(p_{i,j})) \geq v$, $\dim(\tilde{Y}) = n$ and K is a Lie group (Theorem 1.8), and therefore K is discrete. Since the isotropy group K_{p^*} is compact, K_{p^*} is finite. Since $\gamma \in W \subset K_{p^*}$, we may assume the order $o(\gamma) = k < \infty$.

By a standard diagonal argument, we may assume a convergent subsequence,

$$\begin{array}{ccc} (U_d^*, p_{i_j, j}^*, \Lambda_{i_j, j}) & \xrightarrow{GH} & (\tilde{Y}, p^*, K) \\ \downarrow \pi_{i_j, j} & & \downarrow \pi \\ (B_d(p_{i_j, j}), p_{i_j, j}) & \xrightarrow{GH} & (B_d(p), p). \end{array}$$

Let $\gamma_{i, j} \rightarrow \gamma_j$. Observe that for each fixed r_j , $\frac{|p_{i, j}^* \gamma_{i, j}(p_{i, j}^*)|}{r_j} \rightarrow 0$ as $i \rightarrow \infty$. We may assume the above subsequence is chosen so that

$$(2.1.1) \quad \frac{|p_{i_j, j}^* \gamma_{i_j, j}(p_{i_j, j}^*)|}{r_j} \leq j^{-1}.$$

For the sake of simple notation, from now on we will use $i = j = (i_j, j)$.

Let $\gamma_i \in \Lambda_i$ such that $\gamma_i \rightarrow \gamma$. Since for all $m \in \mathbb{Z}$, $\gamma_i^m \rightarrow \gamma^m \in \{\gamma, \dots, \gamma^k = e\}$, and since K is discrete, we conclude that $\langle \gamma_i \rangle \rightarrow \langle \gamma \rangle$ and $o(\gamma_i) = k$ (otherwise, the subgroup, $\langle \gamma_i^k \rangle \rightarrow e$, a contradiction to Lemma 2.3; compare to Remark 2.4).

Observe that if the displacement function of γ_i , $d_{\gamma_i}(z_i^*) = |z_i^* \gamma_i(z_i^*)|$, achieves a minimum at p_i^* , then taking the limit of $(d_{\gamma_i}(p_i^*)^{-1} U_d^*(p_i^*), p_i^*, \langle \gamma_i \rangle)$, it is easy to derive a contradiction (see below). To overcome the trouble that d_{γ_i} may take minimum near the boundary, we claim the following property:

(2.1.2) For each i , there is $q_i^* \in B_{200k \cdot d_{\gamma_i}(p_i^*)}(p_i^*)$ such that $d_{\gamma_i}(q_i^*) \leq d_{\gamma_i}(p_i^*)$ and any $x_i^* \in B_{100k \cdot d_{\gamma_i}(q_i^*)}(q_i^*)$, $d_{\gamma_i}(x_i^*) \geq \frac{1}{100} \cdot d_{\gamma_i}(q_i^*)$.

Assuming (2.1.2), we will derive a contradiction as follows: Since $q_i^* \rightarrow p^*$ and $d_{\gamma_i}(q_i^*) \rightarrow 0$, passing to a subsequence, we may assume

$$(d_{\gamma_i}(q_i^*)^{-1} U_d^*(q_i^*), \langle \gamma_i \rangle) \xrightarrow{GH} (\tilde{Y}', \tilde{q}', \langle \gamma' \rangle)$$

such that $\text{Ric}_{d_{\gamma_i}(q_i^*)^{-1} \tilde{M}_i} \geq -(n-1)d_{\gamma_i}(q_i^*)^2 \rightarrow 0$. Since points in $B_{\frac{d}{4}}(p_i^*)$ are (ϵ_i, r_i) -Reifenberg points, by (2.1.1) we can conclude that \tilde{Y}' is isometric to \mathbb{R}^n . Since $o(\gamma_i) = k$, $o(\gamma') = k$ and thus γ' has a fixed point \tilde{z}' of distance from \tilde{q}' at most $10k$ (\tilde{z}' may be chosen as the center of mass for $\langle \gamma' \rangle(\tilde{q}')$). On the other hand, the choice of q_i^* with the assigned property implies that $d_{\gamma_i} \geq \frac{1}{100}$ on $B_{100k}(q_i^*)$ (after scaling), a contradiction.

Verification of (2.1.2): arguing by contradiction, the failure of (2.1.2) implies that there is $(p_i^*)_1 \in B_{100k \cdot d_{\gamma_i}(p_i^*)}(p_i^*)$ such that $d_{\gamma_i}((p_i^*)_1) < \frac{1}{100} \cdot d_{\gamma_i}(p_i^*)$. Because $(p_i^*)_1$ lies in $B_{200k \cdot d_{\gamma_i}(p_i^*)}(p_i^*)$, there is $(p_i^*)_2 \in B_{100k \cdot d_{\gamma_i}((p_i^*)_1)}((p_i^*)_1)$ such that $d_{\gamma_i}((p_i^*)_2) < \frac{1}{100} \cdot d_{\gamma_i}((p_i^*)_1) < \frac{1}{100^2} d_{\gamma_i}(p_i^*)$. Repeating the process, one gets a sequence of points $(p_i^*)_j$ such that $(p_i^*)_j \in B_{100k \cdot d_{\gamma_i}((p_i^*)_{j-1})}((p_i^*)_{j-1})$ and $d_{\gamma_i}((p_i^*)_j) < \frac{1}{100^j} d_{\gamma_i}(p_i^*)$. Since $(p_i^*)_j \in B_{200k \cdot d_{\gamma_i}(p_i^*)}(p_i^*)$ and the displacement of γ_i has a positive infimum on $B_{200k \cdot d_{\gamma_i}(p_i^*)}(p_i^*)$, this process has to end at a finite step, a contradiction. \square

Remark 2.4. Note that the $\text{vol}(B_\rho(p_i)) \geq v > 0$ is equivalent to that the limit group K is discrete, which guarantees that when $\gamma_i \rightarrow \gamma$, $o(\gamma_i) = o(\gamma)$ for i large. This does not hold if K is not discrete. For instance, let S_i^1 be a sequence of circle subgroup of a maximal torus T^2 of $O(4)$ such that $\text{diam}(T^2/S_i^1) \rightarrow 0$. Let $\mathbb{Z}_{q_i} \subset S_i^1$

such that $\text{diam}(S_i^1/\mathbb{Z}_{q_i}) \rightarrow 0$, where q_i is a prime number. Since T^2 has no fixed point on S_1^3 and $\text{diam}(T^2/S_i^1) \rightarrow 0$, S_i^1 has not fixed point on S_1^3 , and therefore, q_i can be chosen so that \mathbb{Z}_{q_i} acts freely on S_1^3 , and $(S_1^3, \mathbb{Z}_{q_i}) \xrightarrow{GH} (S_1^3, T^2)$. Since T^2 has a circle isotropy subgroup, we may assume $p \in S_1^3$ and $\gamma \in T^2$ of order 2 such that $\gamma(p) = p$. For any $\gamma_i \in \mathbb{Z}_{q_i}$ such that $\gamma_i \rightarrow \gamma$, $o(\gamma_i) = q_i \rightarrow \infty$.

b. Negative curvature and discrete limit isometry groups.

A geometric property of a complete metric of negative Ricci curvature is that if M is compact, then the isometry group is discrete and thus finite ([Bo]). The discreteness does not hold if M is not compact, e.g., $\dim(\text{Isom}(\mathbb{H}^n)) = \frac{n(n+1)}{2}$.

In the proof of Theorem B and Theorem D, we need the following property.

Theorem 2.5. *Assume an equivariant convergent sequence satisfying the following commutative diagram:*

$$\begin{array}{ccc} (\tilde{M}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{GH} & (\tilde{X}, \tilde{p}, G) \\ \downarrow \pi_i & & \downarrow \pi \\ (M_i, p_i) & \xrightarrow{GH} & (X, p), \end{array}$$

where M_i is a compact n -manifold of $\text{diam}(M_i) \leq d$, $\Gamma_i = \pi_1(M_i, p_i)$. If \tilde{X} is isometric to a hyperbolic manifold, then the identity component G_0 is either trivial or not nilpotent.

Let $\phi \in \text{Isom}(\mathbb{H}^n)$. Then ϕ acts on the boundary at infinity of \mathbb{H}^n . From the Poincaré model, by Brouwer fixed point theorem one sees that ϕ has a fixed point on the union of \mathbb{H}^n with its boundary at infinity. Moreover, ϕ satisfies one and only one of the following property: ϕ has a fixed point in \mathbb{H}^n , ϕ has no fixed point in \mathbb{H}^n and a unique fixed point on the boundary at infinity; and ϕ is called elliptic, parabolic and hyperbolic respectively (cf. [Ra]).

Lemma 2.6. *Let M be a complete non-compact hyperbolic manifold. Assume that G is a closed group of isometries, G_0 is nilpotent and M/G is compact. Then*

(2.6.1) G_0 contains no nontrivial compact subgroup.

(2.6.2) If $M = \mathbb{H}^n$, then G_0 contains no hyperbolic element.

Note that in Lemma 2.6, G_0 may not be trivial; e.g., in the half-plane model for \mathbb{H}^n , $\text{Isom}(\mathbb{H}^n)$ contains \mathbb{R}^{n-1} consisting of parabolic elements which fix the same point p_∞ in the boundary at infinity. Let $Z = \langle \mathbb{R}^{n-1}, \gamma \rangle$, where γ is some hyperbolic element which fixes p_∞ . Then \mathbb{H}^n/Z is a circle. Hence, to prove Theorem 2.5 i.e., to rule out parabolic elements in G_0 , we have to use the fact that G is the limiting group of an equivariant convergent sequence.

Proof of Lemma 2.6.

(2.6.1) Since G_0 is nilpotent, G_0 has a unique maximal compact subgroup T^s which is also contained in the center $Z(G_0)$ (Lemma 3, [Wi]). The uniqueness implies that T^s is normal in G . We shall show that $s = 0$.

If $s \geq 1$, let v_1, \dots, v_s denote a basis for the lattice \mathbb{Z}^s ($T^s = \mathbb{R}^s/\mathbb{Z}^s$). Then $H_i = \exp_e tv_i$ is a circle subgroup and $T^s = \prod_{i=1}^s H_i$. The isometric H_i -action

defines a Killing field X_i on M :

$$X_i(x) = \left. \frac{d(H_i(t)(x))}{dt} \right|_{t=0}, \quad x \in M.$$

We define a function on M (cf. [Ro1]),

$$f(x) = \frac{1}{2} \det(g(X_i, X_j))(x), \quad x \in M.$$

Note that $f(x)$ can be viewed as $\frac{1}{2}$ -square of the s -dimensional volume of $T^s(x)$, in particular $f(x)$ is independent of the choice of v_1, \dots, v_s .

Since T^s is normal in G , for $\alpha \in G$, $\alpha(T^s(x)) = T^s(\alpha(x))$ and thus $f(\alpha(x)) = f(x)$. Since f is G -invariant and M/G is compact, we may assume that $f(x)$ achieves a maximum at $y \in M$, and thus $\Delta f(y) \leq 0$. We claim that $f(x)$ satisfies $\Delta f(y) > 0$ at any y such that $f(y) > 0$, and thus a contradiction.

To verify the claim, we first assume that $g_{ij}(y) = g(X_i, X_j)(y) = \delta_{ij}$. Taking any vector fields V_1, \dots, V_{n-s} on a slice of $T^s(y)$ at y such that $g(V_i, V_j)(y) = \delta_{ij}$ and $g(X_i, V_j)(y) = 0$, via the T^s -action we extend V_1, \dots, V_{n-s} to be vector fields on the tube of $T^s(y)$. By construction, $X_1(y), \dots, X_s(y), V_1(y), \dots, V_{n-s}(y)$ is an orthonormal for $T_y M$. For any vector field, Y , by calculation we get

$$\begin{aligned} Y(f)(y) &= Y \left(\frac{1}{2} g_{11} \cdots g_{ss} - \frac{1}{2} \sum_{1 \leq i < j \leq s} g_{ij}^2 g_{11} \cdots \hat{g}_{ii} \cdots \hat{g}_{jj} \cdots g_{ss} + R \right) (y) \\ &= \frac{1}{2} \sum_{i=1}^s g_{11} \cdots Y(g_{ii}) \cdots g_{ss}(y) = \frac{1}{2} \sum_{i=1}^s Y(g_{ii})(y), \\ Y(Y(f))(y) &= \frac{1}{2} \sum_{i=1}^s Y(Y(g_{ii}))(y) + \sum_{1 \leq i < j \leq s} [Y(g_{ii})Y(g_{jj}) - (Y(g_{ij}))^2] (y). \end{aligned}$$

Since $[X_i, X_j] = 0$ and $X_k(g_{ij}) = 0$, by calculation we get

$$\begin{aligned} \Delta f(y) &= \sum_{j=1}^s \text{Hess}_f(X_j, X_j)(y) + \sum_{l=1}^{n-s} \text{Hess}_f(V_l, V_l)(y) \\ &= \frac{1}{2} \sum_{i=1}^s \Delta g_{ii}(y) + \sum_{l=1}^{n-s} \sum_{1 \leq i < j \leq s} [V_l(g_{ii})V_l(g_{jj}) - (V_l(g_{ij}))^2] (y). \end{aligned}$$

Since for any vector fields V, W , any $1 \leq k \leq s$, $g(\nabla_V X_k, W) = -g(\nabla_W X_k, V)$,

$$\begin{cases} \frac{1}{2} \Delta g_{ii}(y) = \sum_{j=1}^s |\nabla_{X_j} X_i|^2(y) + \sum_{l=1}^{n-s} |\nabla_{V_l} X_i|^2(y) - \text{Ric}(X_i, X_i)(y) \\ |\nabla_{X_j} X_i|^2(y) = \sum_{k=1}^s g^2(\nabla_{X_j} X_i, X_k)(y) + \sum_{l=1}^{n-s} g^2(\nabla_{X_j} X_i, V_l)(y) \\ |\nabla_{V_l} X_i|^2(y) = \sum_{k=1}^s g^2(\nabla_{V_l} X_i, X_k)(y) + \sum_{k=1}^{n-s} g^2(\nabla_{V_l} X_i, V_k)(y). \end{cases}$$

Finally,

$$\begin{aligned} \Delta f(y) &= 2 \sum_{l=1}^{n-s} \left[\sum_{i=1}^s g(\nabla_{V_l} X_i, X_i)(y) \right]^2 + \sum_{i,j,k=1}^s g^2(\nabla_{X_j} X_i, X_k)(y) \\ &\quad + \sum_{i=1}^s \sum_{k,l=1}^{n-s} g^2(\nabla_{V_l} X_i, V_k)(y) - \sum_{i=1}^s \text{Ric}(X_i, X_i)(y). \end{aligned}$$

In particular, we conclude that if $f(y) > 0$ i.e., $X_1(y), \dots, X_s(y)$ are linear independent, then $\Delta f(y) > 0$.

In general, at y where $f(y) > 0$ we may choose Killing vector fields, W_1, \dots, W_s , such that $W_1(y), \dots, W_s(y)$ is orthonormal at y , and let $A = (a_{ij})$ be a constant $n \times n$ -matrix such that $W_i(y) = \sum_{j=1}^s a_{ij} X_j(y)$. Then $f(x) = \frac{1}{2} \det(AA^T) \cdot \det(g(W_i, W_j))$, and thus $\Delta f(y) > 0$ at y where $f(y) > 0$.

(2.6.2) Since G_0 is nilpotent, by (2.6.1) we may assume that $Z(G_0) = \mathbb{R}^s$ is not trivial i.e., $s \geq 1$. Assume that $\phi \in Z(G_0)$ is a hyperbolic element i.e., ϕ acts freely on \mathbb{H}^n and has two fixed points on the boundary at infinity. Let $c(t)$ be the unique minimal geodesic connecting the two ϕ -fixed points. Then ϕ preserves $c(t)$, and $c(t)$ is the unique line in \mathbb{H}^n preserved by ϕ (because if a line $\alpha(t)$ is preserved by ϕ , then $c(t)$ and $\alpha(t)$ are preserved by ϕ^2 which fixes the two ends). Since any element in G_0 commutes with ϕ , G_0 preserves $c(t)$, and thus $G_0 = Z(G_0) = \mathbb{R}^1$ such that $c(t)$ is an \mathbb{R}^1 -orbit, which is the unique line \mathbb{R}^1 -orbit. Since \mathbb{R}^1 is normal in G , any element in G preserves $c(t)$, and thus G/\mathbb{R}^1 has a fixed point on $\mathbb{H}^n/\mathbb{R}^1$. Since G/\mathbb{R}^1 is discrete, G/\mathbb{R}^1 is finite. On the other hand, $\mathbb{H}^n/\mathbb{R}^1$ is not compact, because otherwise for $\mathbb{Z} \subset \mathbb{R}^1$, \mathbb{H}^n/\mathbb{Z} is compact hyperbolic manifold on which \mathbb{R}/\mathbb{Z} acts isometrically, a contradiction. Since $\mathbb{H}^n/\mathbb{R}^1$ is not compact and G/\mathbb{R}^1 is finite, $\mathbb{H}^n/G = (\mathbb{H}^n/\mathbb{R}^1)/(G/\mathbb{R}^1)$ is not compact, a contradiction. \square

Proof of Theorem 2.5.

Assume that G_0 is nilpotent. We shall show that $G_0 = e$.

By (2.6.1), we assume that G_0 acts freely on \tilde{X} . We first assume that $\tilde{X} = \mathbb{H}^n$. By (2.6.2), G_0 contains only parabolic elements. Since G_0 is parabolic, in the upper half plane model we see that $G_0(\tilde{p})$ is contained in the horizontal hyperplane \mathbb{R}^{n-1} . Since \mathbb{R}^{n-1} contains no segment, any G_0 -orbit contains no piece of minimal geodesic. We shall derive a contradiction by constructing a sequence of minimal geodesic γ_i on \tilde{M}_i that converges to a minimal geodesic in some G_0 -orbit.

Let $v \in T_e G_0$ be a unit vector, let $\phi = \exp_e v$. Let $t_k = \frac{1}{k} \in [0, 1]$, and let $\phi_k = \exp_e t_k v \in G_0$. From the equivariant convergent commutative diagram,

$$\begin{array}{ccc} (\tilde{M}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{GH} & (\mathbb{H}^n, \tilde{p}, G) \\ \downarrow \pi_i & & \downarrow \pi \\ (M_i, p_i) & \xrightarrow{GH} & (X, p), \end{array}$$

we may assume $\gamma_{i,k} \in \Gamma_i$ such that $\gamma_{i,k} \rightarrow \phi_k$, and thus for any $1 \leq j \leq k$, $\gamma_{i,k}^j \rightarrow \phi_k^j$. Since M_i is compact, we may assume that $p_{i,k}$ is chosen so that $\gamma_{i,k}$ is represented by a close geodesic $c_{i,k}$ at $p_{i,k}$. Consequently, the lifting $\tilde{c}_{i,k}^k$ of $c_{i,k}^k(t)$ at $\tilde{p}_{i,k}$ is a segment that contains a piece of length almost one. Let $\tilde{c}_{i,k}^k \rightarrow \tilde{c}_k \subset \mathbb{H}^n$. Clearly, \tilde{c}_k is a segment. Let $k \rightarrow \infty$ and via a standard diagonal argument we conclude that $\tilde{c}_k \rightarrow \tilde{c}$ is contained in $G_0(\tilde{p})$.

If $\tilde{X} \neq \mathbb{H}^n$, we consider the lifting isometric G_0 -action on \mathbb{H}^n satisfying the following diagram commutes:

$$\begin{array}{ccc} G_0 \times \mathbb{H}^n & \xrightarrow{\tilde{\mu}} & \mathbb{H}^n \\ \downarrow \text{id} \times \pi & & \downarrow \pi \\ G_0 \times \tilde{X} & \xrightarrow{\mu} & \tilde{X}. \end{array}$$

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If $Z(G_0)$ contains a parabolic element, then following the above argument we see that G_0 -orbit in \tilde{X} contains a piece of minimal geodesic, and thus its lifting to \mathbb{H}^n is a piece of minimal geodesic in a G_0 -orbit in \mathbb{H}^n , a contradiction.

If $Z(G_0)$ contains a hyperbolic element, then by the proof of (2.6.2) we see that $G_0 = \mathbb{R}^1$ and G/G_0 fixes a point in \tilde{X}/\mathbb{R}^1 (note that $\pi_1(\tilde{X})$ commutes with the lifting G_0 -action), which contradicts to that \tilde{X}/G is compact. \square

c. The C^0 -convergence.

In the proof of Theorem A, the following C^0 -convergence plays an important role (see the proof of (3.5.2)). Let (M, g) be a compact Riemannian manifold, and let $g(t)$ denote the Ricc flow i.e., the solution of the following PDE ([Ha1]):

$$\frac{\partial g(t)}{\partial t} = -2 \operatorname{Ric}(g(t)), \quad g(0) = g.$$

Theorem 2.7. *Let g_i ($i = 0, 1$) be two Riemannian metrics on a compact n -manifold M such that $\operatorname{Ric}_{g_1} \geq -(n-1)$. Given $\epsilon > 0$, there are constants, $\delta(\epsilon), T = T(n, \epsilon, g_0) > 0$, such that for $0 < \delta \leq \delta(\epsilon)$, if*

$$\operatorname{id}_M : (M, g_1) \rightarrow (M, g_0) \text{ is a } \delta\text{-GHA},$$

then the Ricci flow $g_1(t)$ exists for all $t \in (0, T]$ such that $|g_1(T) - g_0|_{C^0(M)} < \epsilon$.

Note that the existence of $T(n, \epsilon, g_0)$ is a consequence of the Perel'man's pseudo-locality (Theorem 10.1, Corollary 10.2 in [Pe2]). For our purpose, we state it in the following form (cf. Proposition 3.1 in [TW]).

Theorem 2.8. *Given $n, \delta > 0$, there exist constants, $r(n), \epsilon(n), C(n), T(n, \delta) > 0$, such that if a compact n -manifold (M, g) satisfies*

$$\operatorname{Ric}_g \geq -(n-1), \quad d_{GH}(B_r(x), \underline{B}_r^0) < \epsilon(n)r, \quad 0 < r < r(n), \quad x \in M,$$

then the Ricci flow $g(t)$ exists for all $t \in [0, T(n, \delta)]$ and satisfies

$$|\operatorname{Rm}(g(t))|_M \leq \frac{\delta}{t}, \quad \operatorname{vol}(B_{\sqrt{t}}(x, g(t))) \geq C(n)(\sqrt{t})^n.$$

By (1.7.2), a sequence of compact n -manifolds, $M_i \xrightarrow{GH} M$, such that $\operatorname{Ric}_{M_i} \geq -(n-1)$ and M is a Riemannian n -manifold is equivalent to a sequence of Riemannian metrics on M , g_i and g , such that $\operatorname{id}_M : (M, g_i) \rightarrow (M, g)$ is an ϵ_i -GHA, $\epsilon_i \rightarrow 0$.

Corollary 2.9. *Assume a sequence of Riemannian metrics, g_i , and a Riemannian metric g on a compact n -manifold M satisfying*

$$\operatorname{Ric}_{g_i} \geq -(n-1), \quad \operatorname{id}_M : (M, g_i) \rightarrow (M, g) \text{ is an } \epsilon_i\text{-GHA}, \quad \epsilon_i \rightarrow 0.$$

Then passing to a subsequence there is a sequence of Ricci flow solutions of g_i at time $t_i \rightarrow 0$, $g_i(t_i)$, such that $|g_i(t_i) - g|_{C^0(M)} \rightarrow 0$ as $i \rightarrow \infty$.

In the proof of Theorem 2.7, we need the following property for the distance function of $g(t)$, which is due to Bamler-Wilking ([BW]).

Lemma 2.10. *Let the assumption be as in Theorem 2.8. For any $x, y \in M$ with $|xy|_{g(t)} < \sqrt{t}$,*

$$||xy|_g - |xy|_{g(t)}| \leq \Psi(\delta|n)\sqrt{t}.$$

Proof. Because $g(t)$ satisfies that $\text{Ric}_{g(t)} \leq \frac{(n-1)\delta}{t}$, it is known that the function, $|xy|_{g(t)} + 25(n-1)\sqrt{\delta t}$, is monotonically increasing in t (cf. 17. of [Ha2], Corollary 3.26 in [MT]). Consequently, $|xy|_{g(t)} + 25(n-1)\sqrt{\delta t} \geq |xy|_g$.

To prove an opposite inequality, we will assume that $|xy|_{g(t)} < \sqrt{t}$. By Theorem 2.8 and the injectivity radius estimate, we may assume that $\text{injrads}(x, g(t)) \geq \rho\sqrt{t}$ for all x , where ρ is a constant depending on n . Without loss of generality we may assume that $\rho \geq 1$.

Arguing by contradiction, assume some $\sigma > 0$ and given any $\delta_i \rightarrow 0$, there is a sequence of compact n -manifolds (M_i, g_i) , $x_i, y_i \in M_i$ and $t_i \in (0, T(n, \delta_i)]$ such that $|x_i y_i|_{g_i(t_i)} > |x_i y_i|_{g_i} + \sigma\sqrt{t_i}$. Let $d_i = |x_i y_i|_{g_i(t_i)}$. It is easy to check the following relations (assume that $25(n-1)\sqrt{\delta_i} < \frac{\sigma}{4}$) :

$$\begin{cases} B_{d_i - 25(n-1)\sqrt{\delta_i t_i} - \frac{\sigma}{2}\sqrt{t_i}}(x_i, g_i(t_i)) \subset B_{d_i - \frac{\sigma}{2}\sqrt{t_i}}(x_i, g_i) \\ B_{\frac{\sigma}{4}\sqrt{t_i}}(y_i, g_i(t_i)) \subset B_{d_i - \frac{\sigma}{2}\sqrt{t_i}}(x_i, g_i) \end{cases}$$

Let $\ell_i = \frac{d_i}{\sqrt{t_i}}$, and let $s_i = 25(n-1)\sqrt{\delta_i} - \frac{\sigma}{2}$. Then $\sigma < \ell_i \leq 1$ and $s_i \rightarrow -\frac{\sigma}{2}$. Since $B_{\frac{\sigma}{4}\sqrt{t_i}}(y_i, g_i(t_i)) \cap B_{d_i - s_i\sqrt{t_i}}(x_i, g_i(t_i)) = \emptyset$, by ([Ha1]) and Bishop-Gromov volume comparison we derive

$$\begin{aligned} \frac{\text{vol}(B_{\ell_i - \frac{\sigma}{2}}^{-t_i})}{(\sqrt{t_i})^n} &= \text{vol}(B_{d_i - \frac{\sigma}{2}\sqrt{t_i}}^{-1}) \geq \text{vol}_{g_i}(B_{d_i - \frac{\sigma}{2}\sqrt{t_i}}(x_i, g_i)) \\ &\geq \text{vol}_{g_i}(B_{d_i - s_i\sqrt{t_i}}(x_i, g_i(t_i))) + \text{vol}_{g_i}(B_{\frac{\sigma}{4}\sqrt{t_i}}(y_i, g_i(t_i))) \\ &\geq (1 - \Psi(t_i|n)) \left[\text{vol}_{g_i(t_i)}(B_{d_i - s_i\sqrt{t_i}}(x_i, g_i(t_i))) + \text{vol}_{g_i(t_i)}(B_{\frac{\sigma}{4}\sqrt{t_i}}(y_i, g_i(t_i))) \right] \\ &\geq (1 - \Psi(t_i|n)) \left(\frac{\text{vol}(B_{\ell_i - s_i}^{\delta_i})}{(\sqrt{t_i})^n} + \frac{\text{vol}(B_{\frac{\sigma}{4}}^{\delta_i})}{(\sqrt{t_i})^n} \right), \end{aligned}$$

where the last inequality is because $\text{sec}_{t_i^{-1}g_i(t)} \leq \delta_i$ and $\text{injrads}(x_i, g_i(t)) \geq \rho\sqrt{t}$. We may assume that $\ell_i \rightarrow \ell$, $\sigma \leq \ell \leq 1$. As $i \rightarrow \infty$, from the above we conclude that $\text{vol}(B_{\ell - \frac{\sigma}{2}}^0) \geq \text{vol}(B_{\ell - \frac{\sigma}{2}}^0) + \text{vol}(B_{\frac{\sigma}{4}}^0)$, a contradiction. \square

Proof of Theorem 2.7.

Let $\text{id}_M : (M, d_{g_1}) \rightarrow (M, d_{g_0})$ be a δ -GHA, where δ will be specified later. By Theorem 1.6, given $\delta_1 > 0$, we may assume δ small so that (M, g_1) satisfies the conditions of Theorem 2.8 with $\epsilon(n)$ and $r = r(g_0)$, and thus there are constants, $C(n), T = T(n, \delta_1, g_0) > 0$, such that the Ricci flow solution $g_1(t)$ with $t \in (0, T]$ satisfies that

$$|\text{Rm}(g_1(t))|_M \leq \frac{\delta_1}{t}, \quad \text{vol}(B_{\sqrt{t}}(x, g_1(t))) \geq C(n)(\sqrt{t})^n.$$

For all $x \in M$, the re-scaling metric satisfies that

$$|\text{Rm}(T^{-1}g_1(T))|_M \leq \delta_1, \quad \text{vol}(B_1(x, T^{-1}g_1(T))) \geq C(n).$$

By Lemma 2.10,

$$\text{id}_{B_1(x, T^{-1}g_1)} : (B_1(x, T^{-1}g_1), d_{T^{-1}g_1}) \rightarrow (B_1(x, T^{-1}g_1), d_{T^{-1}g_1(T)})$$

is an $\Psi(\delta_1|n)$ -GHA, and thus

$$\text{id}_{B_{\frac{1}{2}}(x, T^{-1}g_0)} : (B_{\frac{1}{2}}(x, T^{-1}g_0), d_{T^{-1}g_1(T)}) \rightarrow (B_{\frac{1}{2}}(x, T^{-1}g_0), d_{T^{-1}g_0})$$

is an $(\Psi(\delta_1|n) + \frac{\delta}{T})$ -GHA. By Cheeger-Gromov $C^{1,\alpha}$ -convergent theorem (cf. [Pet]), we first choose $\delta_1 = \delta_1(\epsilon)$ small so that $\text{id}_{B_1(x, T^{-1}g_0)}$ is an $2\Psi(\delta_1|n)$ -GHA implies that $|T^{-1}g_1(T) - T^{-1}g_0|_{C^{1,\alpha}(B_{\frac{1}{2}}(x, T^{-1}g_0))} < \epsilon$. Note that $\text{id}_{B_1(x, T^{-1}g_0)}$ is an $2\Psi(\delta_1|n)$ -GHA if we choose $\delta(\epsilon) = \Psi(\delta_1|n) \cdot T$. Since the C^0 -norm is scaling invariant, $|g_1(T) - g_0|_{C^0(B_{\frac{\sqrt{T}}{2}}(x, g_0))} < \epsilon$. Because $x \in M$ is arbitrary, $|g_1(T) - g_0|_{C^0(M)} < \epsilon$. \square

3. PROOFS OF THEOREM A-C, THEOREM E AND THEOREM 0.4

Consider a sequence of compact n -manifolds, $M_i \xrightarrow{GH} X$, such that

$$(3.1.1) \quad \text{Ric}_{M_i} \geq (n-1)H, \quad \text{diam}(M_i) \leq d, \quad \text{vol}(B_1(\tilde{p}_i)) \geq v, \quad \frac{\text{vol}(B_\rho(x_i^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon_i \rightarrow 1.$$

From Section 1, subsection b, passing to a subsequence we may assume the following commutative diagram:

$$(3.1.2) \quad \begin{array}{ccc} (\tilde{M}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{GH} & (\tilde{X}, \tilde{p}, G) \\ \downarrow \pi_i & & \downarrow \pi \\ (M_i, p_i) & \xrightarrow{GH} & (X, p), \end{array}$$

where Γ_i denotes the deck transformation group, G is a closed subgroup of $\text{Isom}(\tilde{X})$, which is a Lie group (Theorem 1.8).

Lemma 3.2. *Let \tilde{X} be as in the above. Then \tilde{X} is isometric to Riemannian n -manifold of constant curvature H .*

Proof. For $\tilde{x} \in \tilde{X}$, let $\tilde{x}_i \in \tilde{M}_i$ such that $\tilde{x}_i \rightarrow \tilde{x}$. Let $x_i = \pi_i(\tilde{x}_i)$, and let $\pi_i^* : (U_\rho^*(x_i^*), x_i^*) \rightarrow (B_\rho(x_i), x_i)$ be the Riemannian universal covering. Consider the commutative diagram in (0.17), and by Theorem 1.2,

$$d_{GH}(B_{\frac{\rho}{2}}(x^*), \underline{B}_{\frac{\rho}{2}}^H) = \lim_{i \rightarrow \infty} d_{GH}(B_{\frac{\rho}{2}}(x_i^*), \underline{B}_{\frac{\rho}{2}}^H) \leq \lim_{i \rightarrow \infty} \Psi(\epsilon_i|n, \rho, H) = 0,$$

and thus $B_{\frac{\rho}{2}}(x^*)$ is isometric to $\underline{B}_{\frac{\rho}{2}}^H$. By Bishop-Gromov relative volume comparison, the condition $\text{vol}(B_1(\tilde{p}_i)) \geq v$ implies that for any $\tilde{x}_i \in \tilde{M}_i$, $\text{vol}(B_\rho(\tilde{x}_i)) \geq v(n, \rho, d, H) > 0$. By Corollary 2.2, we can conclude that K acts freely on $B_{\frac{\rho}{4}}(x^*)$, and thus $B_{\frac{\rho}{4}}(\tilde{x})$ is a manifold of constant curvature H . Consequently, \tilde{X} is a manifold of constant curvature H . \square

Corollary 3.3. *Let the assumptions be as in Theorems A-C (resp. $H = 1, -1$ or 0). Then there is $\rho'(n, \rho, d, v) > 0$ such that the Riemannian universal covering \tilde{M} satisfies*

$$\frac{\text{vol}(B_{\rho'}(\tilde{x}))}{\text{vol}(\underline{B}_{\rho'}^H)} \geq 1 - \Psi(\epsilon|n, \rho, d, v), \quad \tilde{x} \in \tilde{M}.$$

Proof. Arguing by contradiction, assume $\rho_k \rightarrow 0$ such that for each ρ_k there is $\epsilon(\rho_k) > 0$ and a sequence $M_{i,k}$ such that

$$\frac{\text{vol}(B_{\rho}(x_{i,k}^*))}{\text{vol}(\underline{B}_{\rho}^H)} \geq 1 - \epsilon_i \rightarrow 1, \quad \forall x_{i,k} \in M_{i,k},$$

and there is $\tilde{q}_{i,k} \in \tilde{M}_{i,k}$ such that

$$(3.3.1) \quad \frac{\text{vol}(B_{\rho_k}(\tilde{q}_{i,k}))}{\text{vol}(\underline{B}_{\rho_k}^H)} < 1 - \epsilon(\rho_k), \quad \forall i.$$

Passing to a subsequence, we may assume $(\tilde{M}_{i,k}, \tilde{q}_{i,k}) \xrightarrow{GH} (\tilde{X}_k, \tilde{q}_k)$. By Lemma 3.2, \tilde{X}_k is isometric to space form of constant curvature H and $\text{vol}(B_1(\tilde{q}_k)) \geq v'(n, \rho, d, v) > 0$ (Theorem 1.6). By Cheeger's injectivity estimate, we may assume that $\text{inrad}(\tilde{q}_k) \geq \rho'(n, \rho, d, v) > 0$. For fixed $\rho_k < \frac{\rho'}{2}$, by Theorem 1.6 we have that $\text{vol}(B_{\rho_k}(\tilde{q}_{i,k})) \rightarrow \text{vol}(\underline{B}_{\rho_k}^H)$, a contradiction to (3.3.1). \square

a. Proofs of Theorem A-C.

Consider a sequence in (3.1.1) and (3.1.2) with $H = 1$, and thus \tilde{X} is isometric to S_1^n (Lemma 3.2, Theorem 1.7). In the proof of Theorem A, we need the following result in [MRW].

Lemma 3.4. *Let $M_i \xrightarrow{GH} X$ be a sequence of compact n -manifolds satisfying*

$$\text{Ric}_{M_i} \geq -(n-1), \quad \text{diam}(M_i) \leq d, \quad \text{vol}(B_1(\tilde{p}_i)) \geq v > 0,$$

and the commutative diagram (3.1.2). If Γ_i is finite, then for i large, there is an injective homomorphism, $\phi_i : \Gamma_i \rightarrow G$, which is also an ϵ_i -GHA with $\epsilon_i \rightarrow 0$.

Note that Lemma 3.4 was originally stated in [MRW] under the condition that $\text{sec}_{M_i} \geq -1$. Because the sectional curvature condition was used only to conclude that a limiting group is a Lie group, by Theorem 1.8 Lemma 3.4 is valid when ‘ $\text{sec}_{M_i} \geq -1$ ’ is replaced by ‘ $\text{Ric}_{M_i} \geq -(n-1)$ ’.

Let $\phi_i : \Gamma_i \rightarrow G$ be as in Lemma 3.4. By Theorem 1.7, we may assume a diffeomorphism, $\tilde{h}_i : \tilde{M}_i \rightarrow S_1^n$, such that (\tilde{h}_i, ϕ_i) is also an ϵ_i -equivariant GHA i.e., for all $\tilde{x}_i \in \tilde{M}_i$ and $\gamma_i \in \Gamma_i$,

$$|\tilde{h}_i(\tilde{x}_i)[\phi_i(\gamma_i)\tilde{h}_i(\gamma_i^{-1}(\tilde{x}_i))]| < \epsilon_i.$$

Note that via \tilde{h}_i , Γ_i acts freely on \tilde{X} : $\gamma_i(\tilde{x}) = \tilde{h}_i(\gamma_i(\tilde{h}_i^{-1}(\tilde{x})))$ for $\tilde{x} \in \tilde{X}$ and $\gamma_i \in \Gamma_i$. We shall use $\Gamma_i(\tilde{h}_i)$ to denote the Γ_i -action on \tilde{X} via \tilde{h}_i .

Theorem 3.5. *Let M_i be a sequence of compact n -manifolds satisfying*

$$\text{Ric}_{M_i} \geq (n-1), \quad \frac{\text{vol}_\rho(B_\rho(\tilde{x}_i))}{\text{vol}(\underline{B}_\rho^1)} \geq 1 - \epsilon_i \rightarrow 1, \quad \tilde{x}_i \in \tilde{M}_i$$

and the commutative diagram (3.1.2). Then for i large,

(3.5.1) $\phi_i(\Gamma_i)$ acts freely on S_1^n .

(3.5.2) The $\Gamma_i(\tilde{h}_i)$ -action and the $\phi_i(\Gamma_i)$ -action on S_1^n are conjugate.

Proof. (3.5.1) If $e \neq \gamma_i \in \Gamma_i$, $\tilde{y} \in S_1^n$ such that $\phi_i(\gamma_i)(\tilde{y}) = \tilde{y}$, then $\langle \gamma_i \rangle \rightarrow \Lambda \neq e$ (Lemma 2.3) and $\Lambda(\tilde{y}) = \tilde{y}$. Without loss of generality, we may assume \tilde{y} is chosen such that $\tilde{x}_i \rightarrow \tilde{y}$ and the displacement of γ_i achieves a minimum at \tilde{x}_i . Since $\langle \gamma_i \rangle(\tilde{x}_i) \xrightarrow{GH} \Lambda(\tilde{y}) = \tilde{y}$, $r_i = \text{diam}(\langle \gamma_i \rangle(\tilde{x}_i)) \rightarrow 0$. Consider the rescaling sequence,

$$(r_i^{-1}\tilde{M}_i, \tilde{x}_i, \langle \gamma_i \rangle) \xrightarrow{GH} (\mathbb{R}^n, v, K).$$

Since $\text{diam } K(v) = 1$, K is compact. Then K has a fixed point, say 0, and let $\tilde{z}_i \in r_i^{-1}\tilde{M}_i$ such that $\tilde{z}_i \rightarrow 0$. Then $\langle \gamma_i \rangle(\tilde{z}_i) \rightarrow K(0) = 0$. This is not possible, because

$$\text{diam}(\langle \gamma_i \rangle(\tilde{z}_i)) \geq \text{diam}(\langle \gamma_i \rangle(\tilde{x}_i)) = 1,$$

a contradiction.

(3.5.2) Let \tilde{g}_i denote the pullback metric on S^n by \tilde{h}_i^{-1} . Then the identity map, $\text{id}_{S^n} : (S^n, \tilde{g}_i, \Gamma_i(\tilde{h}_i)) \rightarrow (S^n, \underline{g}^1, \phi_i(\Gamma_i))$, is an ϵ_i -equivariant GHA. A natural way to obtain an equivariant map is via the method of center of mass with respect to \underline{g}^1 : fixing $\tilde{x} \in S^n$, let $A(\tilde{x}) = \{\phi_i(\gamma_i)^{-1}(\gamma_i(\tilde{x})), \gamma_i \in \Gamma_i(\tilde{h}_i)\}$. Since $A(\tilde{x}) \subset B_{\frac{\pi}{4}}(\tilde{x})$, $A(\tilde{x})$ has a center of mass, say \tilde{y} . We then define $\tilde{f}_i : S_1^n \rightarrow S_1^n$ by $\tilde{f}_i(\tilde{x}) = \tilde{y}$. Then \tilde{f}_i is a differentiable map satisfying that $\tilde{f}_i(\gamma_i(\tilde{x})) = \phi_i(\gamma_i)(\tilde{f}_i(\tilde{x}))$.

According to [GK], \tilde{f}_i is a diffeomorphism if the two actions are ϵ -close in C^1 -norm i.e., $\max\{|\tilde{x}\phi_i(\gamma_i)^{-1}\gamma_i(\tilde{x})|_{\underline{g}^1}, \tilde{x} \in S^n\} < \epsilon$ and $|d(\phi_i(\gamma_i)^{-1}\gamma_i)(X) - \mathbb{P}(X)|_{\underline{g}^1} < \Psi(\epsilon)$ for all $\gamma_i \in \Gamma_i(\tilde{h}_i)$ and $|X|_{\underline{g}^1} = 1$, where \mathbb{P} denotes the \underline{g}^1 -parallel translation along the unique minimal geodesic joining \tilde{x} and $\phi_i(\gamma_i)^{-1}\gamma_i(\tilde{x})$ and $\epsilon > 0$ is a constant determined by \underline{g}^1 .

Given $\epsilon > 0$, by Theorem 2.7 we may assume that $\text{id}_{S^n} : (S^n, \tilde{g}_i(T)) \rightarrow (S^n, \underline{g}^1)$ is an $\delta(\epsilon)$ -GHA for i large, where $T = T(n, \epsilon, \underline{g}^1) > 0$ such that $|T^{-1}\tilde{g}_i(T) - T^{-1}\underline{g}^1|_{C^{1,\alpha}(B_{\frac{1}{2}}(\tilde{x}, T^{-1}\underline{g}^1))} < \epsilon$ (see the end of proof of Theorem 2.7). Consequently, restricting to $B_{\frac{1}{2}}(\tilde{x}, T^{-1}\underline{g}^1)$, exponential maps of $T^{-1}g_i(T)$ and $T^{-1}\underline{g}^1$ are C^α -close, and therefore the $\Gamma_i(\tilde{h}_i)$ and $\phi_i(\Gamma_i)$ -actions are ϵ -close in C^1 -norm. Since $\epsilon > 0$ is arbitrary, the desired conclusion follows. \square

Proof of Theorem A.

Arguing by contradiction, assume a sequence, $M_i \xrightarrow{GH} X$, satisfying (3.1.1) and (3.1.2) for $H = 1$ such that M_i is not diffeomorphic to any spherical n -space form. By Lemma 3.2, \tilde{X} is isometric to spherical space form. By Theorem 1.7, \tilde{X} is diffeomorphic to \tilde{M}_i which is simply connected, and therefore $\tilde{X} = S_1^n$. By (3.5.1) and (3.5.2), $M_i = \tilde{M}_i/\Gamma_i$ is diffeomorphic to $S_1^n/\phi_i(\Gamma_i)$, a contradiction. \square

Proof of Theorem B.

Arguing by contradiction, assume a sequence, $M_i \xrightarrow{GH} X$, satisfying (3.1.1) and (3.1.2) for $H = -1$ such that M_i is not diffeomorphic to any hyperbolic n -manifold. By Lemma 3.2, \tilde{X} is isometric to a hyperbolic n -manifold (we do not yet know that \tilde{X} is simply connected). We claim that there is a constant $c(n, \rho, d, v) > 0$ such that $\text{vol}(M_i) \geq c(n, \rho, d, v)$. Consequently, G is discrete. By Corollary 3.3 we are able to apply Theorem 2.1 and conclude that G acts freely on \tilde{X} and thus $X = \tilde{X}/G$ is isometric to a hyperbolic n -manifold. By Theorem 1.7, M_i is diffeomorphic to \tilde{X}/G , a contradiction.

If the above claim fails, then $\dim(X) < n$ and thus $\dim(G_0) > 0$. By Lemma 1.13 there is $\epsilon > 0$ such that $\Gamma_i^\epsilon \rightarrow G_0$. By Theorem 1.9, Γ_i^ϵ has a nilpotent subgroup of bounded index, and thus $G_0 \neq e$ is nilpotent, a contradiction to Theorem 2.5. \square

Proof of Theorem C.

Arguing by contradiction, we may assume a sequence $M_i \xrightarrow{GH} X$ satisfying (3.1.1) and (3.1.2) for $H = 0$ and M_i is not flat. By Lemma 3.2, \tilde{X} is a flat manifold, and thus $\tilde{X} = \mathbb{R}^k \times F^{n-k}$ and F^{n-k} is a compact flat manifold. On the other hand, by Splitting theorem of Cheeger-Gromoll, $\tilde{M}_i = \mathbb{R}^{k_i} \times N_i$, where N_i is a compact simply connected manifold of non-negative Ricci curvature.

We claim that $\text{diam}(N_i) \leq D(n)$ a constant depending on n , and without loss of generality we may further assume that $\text{diam}(F^{n-k}) \leq D(n)$. Consequently, for any $R > D(n)$ and i large, $B_R(\tilde{p}_i)$ is simply connected and is diffeomorphic to $B_R(\tilde{p})$ (Theorem 1.7), which implies that $n - k = 0$, and thus N_i is a point i.e., M_i is a flat manifold, a contradiction.

Assuming that $\text{diam}(N_i) = r_i \rightarrow \infty$, passing to a subsequence we may assume

$$\begin{array}{ccc} (r_i^{-1}\mathbb{R}^{k_i} \times N_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{GH} & (\mathbb{R}^k \times N, \tilde{p}', G') \\ \downarrow \pi_i & & \downarrow \pi \\ (r_i^{-1}M_i, p_i) & \xrightarrow{GH} & p, \end{array}$$

where N is a compact length space of diameter 1. Note that $G' = G'_0$ is a nilpotent group (Theorem 1.9) acting effectively and transitively on $\mathbb{R}^k \times N$. Consequently, N is a s -torus ($s \geq 1$). Since $r_i^{-1}N_i \xrightarrow{GH} N = T^s$, there is an onto map from $\pi_1(N_i) \rightarrow \pi_1(T^s)$ (cf. [Tu]), a contradiction⁵. \square

b. Proof of Theorem E.

Lemma 3.6. *Given $n, \rho > 0$, there exists a constant $\epsilon(n, \rho) > 0$ such that for any $0 < \epsilon < \epsilon(n, \rho)$, if a compact Einstein n -manifold M of Ricci curvature $\equiv H$ satisfies*

$$\frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon, \quad \forall x \in M,$$

then the sectional curvature is almost constant i.e.,

$$H - \Psi(\epsilon|n, \rho) \leq \sec_M \leq H + \Psi(\epsilon|n, \rho).$$

⁵The proof of $\text{diam}(N_i) \leq D(n)$ was due to J. Pan.

Proof. Arguing by contradiction, assuming a sequence $\epsilon_i \rightarrow 0$ and a sequence of Einstein n -manifolds M_i which satisfy the conditions of Lemma 3.6 with respect to ϵ_i , but there are $p_i \in M_i$ and a plane $\Sigma_i \subset T_{p_i} M_i$ such that $|\sec(\Sigma_i) - H| \geq \delta > 0$.

By Theorem 1.2, passing to a subsequence we may assume that $B_\rho(p_i^*) \xrightarrow{GH} \underline{B}_\rho^H$. Since for i large, $B_{\frac{\rho}{2}}(p_i^*)$ is diffeomorphic to $\underline{B}_{\frac{\rho}{2}}^H$ (compare to (1.7.2)), we may identify the sequence as a sequence of metrics d_i^* on $\underline{B}_{\frac{\rho}{2}}^H$ that converges to \underline{d}^H . Since the lifting metrics g_i^* on $B_\rho(p_i^*)$ is Einstein, passing to a subsequence we may assume that $g_i^* \xrightarrow{C^k} \underline{g}^H$ for any $k < \infty$ ([Ch]). In particular, $\sec_{g_i^*}|_{B_{\frac{\rho}{2}}(p_i^*)} \rightarrow H$ i.e.,

$$H - \Psi(\epsilon_i|n, \rho) \leq \sec_{B_{\frac{\rho}{2}}(p_i)} \leq H + \Psi(\epsilon_i|n, \rho),$$

a contradiction. \square

Proof of Theorem E.

By Lemma 3.6, M_i has almost constant sectional curvature H .

Case 1. Assume $H = -1$. Since M has bounded negative sectional curvature, by Heintze-Margulis lemma ([He]) we may assume $\text{vol}(M) \geq v(n) > 0$. By now the desired conclusion follows from Theorem B.

Case 2. Assume $H = 0$. Then M is almost flat, and thus by Gromov's almost flat manifolds theorem \tilde{M} is contractible ([Gr]). By Cheeger-Gromoll Splitting theorem it follows that M is flat.

Case 3. Assume $H = 1$. First, since the curvature is almost one, the classical $1/4$ -pinched injectivity radius estimate implies that \tilde{M} has injectivity radius $> \frac{\pi}{2}$. By now the desired conclusion follows from Theorem A. \square

Remark 3.7. In a forthcoming paper [CRX], we will generalize Theorem E to manifolds with bounded Ricci curvature.

c. Proof of Theorem 0.4.

We first extend Theorem C to a limit space.

Lemma 3.7. *Given $n, \rho, v > 0$, there is $\epsilon(n, \rho, v) > 0$ such that if X is the limit space of a sequence of compact n -manifolds M_i such that*

$$\text{Ric}_{M_i} \geq -(n-1)\delta_i \rightarrow 0, \text{diam}(M_i) = 1, \text{vol}(M_i) \geq v, \frac{\text{vol}(B_\rho(x_i^*))}{\text{vol}(\underline{B}_\rho^0)} \geq 1 - \epsilon(n, \rho, v),$$

then X is isometric to a flat manifold.

Proof. Arguing by contradiction, assume a sequence X_i such that X_i is not isometric to any flat manifold, and X_i is the limit of a sequence of compact n -manifolds, $M_{ij} \xrightarrow{GH} X_i$, as $j \rightarrow \infty$, and M_{ij} satisfies the conditions in Lemma 3.7 with $\delta_{ij} \rightarrow 0$ and $\epsilon_i \rightarrow 0$. Passing to a subsequence, we may assume that $X_i \xrightarrow{GH} X$, and by a standard diagonal argument we may assume a sequence, $M_{ij(i)} \xrightarrow{GH} X$. By Theorem 1.2, passing to a subsequence we may assume $B_{\frac{\rho}{2}}(x_{ij(i)}^*) \xrightarrow{GH} \underline{B}_{\frac{\rho}{2}}^0$. By Corollary 2.2, if $x_{ij(i)} \rightarrow x$, then a small ball around x is isometric to an Euclidean ball, and thus X is a flat n -manifold.

Since X_i is homeomorphic to $M_{ij(i)}$ ((1.7.1)), which, by the same reason, is diffeomorphic to X , X_i is homeomorphic to X . Since $\delta_{ij} \rightarrow 0$ as $j \rightarrow \infty$, \tilde{X}_i satisfies the Splitting property ([CC]), and thus \tilde{X}_i is isometric to $\mathbb{R}^{k_i} \times N_i$ and N_i is compact simply connected topological manifold. Since X is flat, $\tilde{X}_i = \mathbb{R}^n$ and thus X_i is flat, a contradiction. \square

Proof of Theorem 0.4.

Arguing by contradiction, assume $\delta_i \rightarrow 0$ and a sequence of compact n -manifolds, $M_i \xrightarrow{GH} X$, such that M_i is not diffeomorphic to any flat manifold and

$$\text{Ric}_{M_i} \geq -(n-1)\delta_i, \quad 1 \geq \text{diam}(M_i), \quad \text{vol}(M_i) \geq v, \quad \frac{\text{vol}(B_\rho(x_i^*))}{\text{vol}(\underline{B}_\rho^0)} \geq 1 - \epsilon(n, \rho, v),$$

where $\epsilon(n, \rho, v)$ is from Lemma 3.7. By Lemma 3.7, X is isometric to a flat manifold, and by Theorem 1.7 for i large M_i is diffeomorphic to X , a contradiction. \square

4. PROOF OF THEOREM D BY ASSUMING THEOREM 1.4.

Using Theorem 1.4, we will establish the following result.

Theorem 4.1. *Let $M_i \xrightarrow{GH} X$ be a sequence of compact n -manifolds such that*

$$\text{Ric}_{M_i} \geq -(n-1), \quad \text{diam}(M_i) \leq d, \quad h(M_i) \geq n-1-\epsilon_i \rightarrow n-1.$$

Then the sequence of Riemannian universal covering spaces, $(\tilde{M}_i, \tilde{p}_i) \xrightarrow{GH} (\mathbb{H}^n, o)$.

Proof of Theorem D by assuming Theorem 4.1.

Arguing by contradiction, assume a sequence of compact n -manifolds, $M_i \xrightarrow{GH} X$, as in Theorem 4.1 such that (3.1.2) holds and M_i is not diffeomorphic or not close to any hyperbolic manifold. By Theorem 4.1, \tilde{X} is isomorphic to \mathbb{H}^n . By applying Theorem 1.6 on \tilde{M}_i , it is clear that M_i satisfies the conditions of Theorem B, a contradiction. \square

Our proof of Theorem 4.1 is divided into two steps: we first show that \tilde{X} is isometric to \mathbb{H}^k , $1 \leq k \leq n$ (Lemma 4.4). Then we show that $\lim_{i \rightarrow \infty} h(M_i) = k-1$ (Theorem 4.6), and thus conclude that $k = n$.

To apply Theorem 1.4, we will need to extend an observation in [Li]: if a compact n -manifold of $\text{Ric}_M \geq -(n-1)$ has the maximal volume entropy $n-1$, then there is a sequence, $r_i \rightarrow \infty$, such that for any $\epsilon > 0$, (1.5.1) is satisfied for $L = r_i$ when i large.

Lemma 4.2. *Let \tilde{M} be a complete Riemannian n -manifold with $\text{Ric}_{\tilde{M}} \geq -(n-1)$ and*

$$h(\tilde{M}) = \limsup_{r \rightarrow \infty} \frac{1}{r} \ln \text{vol}(B_r(\tilde{p})) \geq n-1-\epsilon.$$

Then fixing $R > 0$ and $\tilde{p} \in \tilde{M}$, there exists a sequence $r_i \rightarrow \infty$, such that

$$(4.2.1) \quad \lim_{i \rightarrow \infty} \frac{\text{vol}(\partial B_{r_i+50R}(\tilde{p}))}{\text{vol}(\partial B_{r_i-50R}(\tilde{p}))} \geq e^{100R(n-1-\epsilon)},$$

where $e^{100R(n-1)}$ is the limit ratio of the same type in \mathbb{H}^n .

Proof. Arguing by contradiction, we may assume sufficiently small $\epsilon_0 > 0$ and $r_0 > 100R$ such that for any $r \geq r_0$,

$$\lim_{i \rightarrow \infty} \frac{\text{vol}(\partial B_{r+50R}(\tilde{p}))}{\text{vol}(\partial B_{r-50R}(\tilde{p}))} < (1 - \epsilon_0) \cdot e^{100R(n-1-\epsilon)}.$$

Then by iteration

$$\begin{aligned} \text{vol}(\partial B_r(\tilde{p})) &\leq (1 - \epsilon_0) e^{100R(n-1-\epsilon)} \text{vol}(\partial B_{r-100R}(\tilde{p})) \\ &\leq C(n, r_0, R) \cdot \left((1 - \epsilon_0) e^{100R(n-1-\epsilon)} \right)^{\frac{r-r_0}{100R}}. \end{aligned}$$

Thus,

$$\begin{aligned} h(\tilde{M}) &= \limsup_{r \rightarrow \infty} \frac{1}{r} \ln(\text{vol}(B_r(\tilde{p}))) \\ &= \limsup_{r \rightarrow \infty} \frac{1}{r} \ln \left(\int_0^r \text{vol}(\partial B_u(\tilde{p})) du \right) \\ &\leq \limsup_{r \rightarrow \infty} \frac{1}{r} \ln \left(\int_{r_0}^r C(n, r_0, R) \cdot \left((1 - \epsilon_0) e^{100R(n-1-\epsilon)} \right)^{\frac{u-r_0}{100R}} du + \text{vol}(B_{r_0}(\tilde{p})) \right) \\ &= n - 1 - \epsilon + \frac{\ln(1 - \epsilon_0)}{100R} \\ &< n - 1 - \epsilon, \end{aligned}$$

a contradiction. \square

By Lemma 4.2, we are able to apply Theorem 1.4 to prove the following:

Lemma 4.3. *Let M be a compact Riemannian n -manifold such that*

$$\text{Ric}_M \geq -(n-1), \quad h(M) \geq n-1-\epsilon.$$

For $R > 0$, and any $\tilde{p} \in \tilde{M}$, there is a connected length metric space X such that

$$d_{GH}(B_R(\tilde{p}), B_R((0, y))) \leq \Psi(\epsilon|n, R),$$

where $B_R((0, y))$ is a metric ball in a warped product space $\mathbb{R}^1 \times_{e^s} Y$.

Proof. Let $R > 50 \text{diam}(M)$. By Lemma 4.2, there is $r_i \rightarrow \infty$ such that (4.2.1) holds. Because

$$\lim_{r \rightarrow \infty} \frac{\text{vol}(\partial \underline{B}_{r+50R}^{-1})}{\text{vol}(\partial \underline{B}_{r-50R}^{-1})} = e^{100R(n-1)},$$

condition (1.5.1) is equivalent to (4.2.1) for $L = r_i > 2R$. By Theorem 1.4, for large i , $A_{r_i-50R, r_i+50R}(\tilde{p})$ contains a ball, $B_{2R}(\tilde{q})$, such that

$$d_{GH}(B_{2R}(\tilde{q}), B_{2R}((0, y))) \leq \Psi(\epsilon|n, d, R),$$

where $B_{2R}((0, y))$ is a metric ball in a warped product space $\mathbb{R}^1 \times_{e^s} Y$, and Y is a length metric space. Because $R > 50 \text{diam}(M)$, we may assume that $B_{\text{diam}(M)}(\tilde{q})$ contains a point $\tilde{p}' = \gamma(\tilde{p})$, where γ is a deck transformation of \tilde{M} . Then $B_R(\tilde{p}') \subset B_{2R}(\tilde{q})$, and this completes the proof. \square

Lemma 4.4. *Let the assumptions be as in Theorem 4.1. Then by passing to a subsequence, $(\tilde{M}_i, \tilde{p}_i) \xrightarrow{GH} (\mathbb{H}^k, o)$ ($k \leq n$).*

Remark 4.5. Observe that in Lemma 4.4, if $M_i = M$, then $\tilde{M} = \mathbb{H}^n$, and thus M is a hyperbolic manifold. This gives a different proof of Theorem 0.3, which does not rely on [LiW] (cf. [LW1], [Li]).

Proof of Lemma 4.4.

Passing to a subsequence, assume that (3.1.2) holds. Fixing any $R > 0$, by Lemma 4.3,

$$\begin{aligned} d_{GH}(B_R(\tilde{p}), B_R((0, y_i))) &\leq d_{GH}(B_R(\tilde{p}), B_R(\tilde{p}_i)) + d_{GH}(B_R(\tilde{p}_i), B_R((0, y_i))) \\ &\leq \Psi(\epsilon_i | n, d, R), \end{aligned}$$

where $B_R((0, y_i))$ is a metric ball in a warped product space $\mathbb{R}^1 \times_{e^s} Y_i$. Note that $(\mathbb{R}^1 \times_{e^s} Y_i, (0, y_i)) \xrightarrow{GH} (\mathbb{R}^1 \times_{e^s} Y, (0, y))$. Since R is arbitrary, we conclude that (\tilde{X}, \tilde{p}) is isometric to $(\mathbb{R}^1 \times_{e^s} Y, (0, y))$.

Since \tilde{X} is a limit of manifolds of Ricci curvature bounded below, regular points in \tilde{X} are dense; a point is regular if the tangent is unique and isometric to \mathbb{R}^k for some $k \leq n$. Without loss of generality, we may assume that \tilde{p} is a regular point, and thus $\lim_{t \rightarrow \infty} (e^t Y, y) = (\mathbb{R}^{k-1}, 0)$. Via reparametrization of $s' = s - t$,

$$\begin{aligned} \lim_{t \rightarrow \infty} (\mathbb{R}^1 \times_{e^s} Y, (t, y)) &= \lim_{t \rightarrow \infty} (\mathbb{R}^1 \times_{e^{s'}} e^t Y, (0, y)) \\ &= (\mathbb{R}^1 \times_{e^s} \mathbb{R}^{k-1}, o) = (\mathbb{H}^k, o). \end{aligned}$$

Since \tilde{X}/G is compact, for any $t \in \mathbb{R}^1$, there is $\gamma_t \in G$ such that $d(\gamma_t(\tilde{p}), (t, y)) \leq \text{diam}(X) \leq d$.

$$(\tilde{X}, \tilde{p}) = \lim_{t \rightarrow \infty} (\tilde{X}, \gamma_t(\tilde{p})) = (\mathbb{H}^k, o).$$

□

Theorem 4.6. *Let $M_i \xrightarrow{GH} X$ be a sequence satisfying*

$$\text{Ric}_{M_i} \geq -(n-1), \quad \text{diam}(M_i) \leq d,$$

and the following commutative diagram,

$$\begin{array}{ccc} (\tilde{M}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{GH} & (\mathbb{H}^k, \tilde{p}, G) \\ \downarrow \pi_i & & \downarrow \pi \\ (M_i, p_i) & \xrightarrow{GH} & (X, p) \end{array}$$

Then $\lim_{i \rightarrow \infty} h(M_i) = k - 1$.

Note that Theorem 4.1 follows from Lemma 4.4 and Theorem 4.6.

By Section 1.b, the commutative diagram in Theorem 4.6 yields the following commutative diagram:

$$\begin{array}{ccc} (\tilde{M}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{GH} & (\mathbb{H}^k, \tilde{p}, G) \\ \downarrow \hat{\pi}_i & & \downarrow \hat{\pi} \\ (\hat{M}_i, \hat{p}_i, \hat{\Gamma}_i) & \xrightarrow{GH} & (\hat{X}, \hat{p}, \hat{G}) \\ \downarrow \bar{\pi}_i & & \downarrow \bar{\pi} \\ (M_i, p_i) & \xrightarrow{GH} & (X, p), \end{array}$$

where $\Gamma_i \cong \pi_1(M_i)$, $\hat{M}_i = \tilde{M}_i/\Gamma_i^\epsilon$, $\hat{X} = \mathbb{H}^k/G_0$, and $\hat{\Gamma}_i = \Gamma_i/\Gamma_i^\epsilon \cong G/G_0 = \hat{G}$. By Lemma 1.13, we may assume an isomorphism $\hat{\phi}_i : \hat{\Gamma}_i \rightarrow \hat{G}$ such that for any $R > 0$ $(\hat{h}_i, \hat{\phi}_i, \hat{\phi}_i^{-1})$ is an ϵ_i -equivariant GHA on $(B_R(\hat{p}_i), \hat{\Gamma}_i(R))$ (see (1.10)). As seen in the proof of Theorem B, G_0 is nilpotent (Theorem 1.9) and thus $G_0 = e$ (Theorem 2.5), and thus $\hat{G} = G/G_0 = G$ is discrete.

Lemma 4.7. *Let the assumptions be as in Theorem 4.6. Then for i large, there is a map $\hat{f}_i : (\hat{M}_i, \hat{p}_i) \rightarrow (\hat{X}, \hat{p})$ such that*

(4.7.1) \hat{f}_i is $\hat{\Gamma}_i$ -conjugate, i.e., $\hat{f}_i(\gamma_i(\hat{q}_i)) = \hat{\phi}_i(\gamma_i)\hat{f}_i(\hat{q}_i)$;

(4.7.2) for any $R > 0$, $\hat{f}_i|_{B_R(\hat{p}_i)} : B_R(\hat{p}_i) \rightarrow B_{(1+\frac{\epsilon_i}{60d})R}(\hat{f}_i(\hat{p}_i))$ is an $\frac{R}{10d}\epsilon_i$ -GHA.

Proof. Fix any $R_0 > 50d$. Let $\hat{h}_i : (B_{\frac{1}{\epsilon_i}}(\hat{p}_i), \hat{p}_i) \rightarrow (\hat{X}, \hat{p})$ be an ϵ_i -equivariant GHA with respect to $\hat{\phi}_i : \hat{\Gamma}_i \rightarrow G$. We shall apply center of mass method to modify \hat{h}_i to obtain \hat{f}_i . Let $\delta = \min\{\text{convex radius at } x \in B_{2R_0}(\hat{p})\} > 0$. For i large,

$$E(\hat{x}_i) = \left\{ \hat{\phi}_i(\hat{\gamma}_i)^{-1}(h_i(\hat{\gamma}_i(\hat{x}_i))), \hat{\gamma}_i \in \hat{\Gamma}_i(R_0) \right\} \subset B_\delta(\hat{h}_i(\hat{x}_i)).$$

Let \hat{z}_i be the unique center of mass of $E(\hat{x}_i)$. We then define $\hat{h}'_i(\hat{x}_i) = \hat{z}_i$. It is clear that $\hat{h}'_i : B_{R_0}(\hat{p}_i) \rightarrow B_{R_0+2\epsilon_i}(\hat{h}'_i(\hat{p}_i))$ is a $\hat{\Gamma}_i(R_0)$ -conjugate $2\epsilon_i$ -GHA.

We then extend \hat{h}'_i to a map $\hat{f}_i : \hat{M}_i \rightarrow \hat{X}$ as follows. For any $\hat{y}_i \in \hat{M}_i$, there is an element $\hat{\alpha}_i \in \hat{\Gamma}_i$ such that $|\hat{\alpha}_i(\hat{y}_i)\hat{p}_i| \leq d$. We define

$$\hat{f}_i(\hat{y}_i) = \hat{\phi}_i(\hat{\alpha}_i)^{-1}(\hat{h}'_i(\hat{\alpha}_i(\hat{y}_i))).$$

Note that $\hat{f}_i(\hat{y}_i)$ is independent of the choice of $\hat{\alpha}_i$, and thus \hat{f}_i is $\hat{\Gamma}_i$ -conjugate; if $\hat{\beta}_i \in \hat{\Gamma}_i$ satisfies $|\hat{\beta}_i(\hat{y}_i)\hat{p}_i| \leq d$, let $\hat{\gamma}_i = \hat{\alpha}_i \cdot \hat{\beta}_i^{-1} \in \hat{\Gamma}_i(R_0)$. Since \hat{h}'_i is $\hat{\Gamma}_i(R_0)$ -conjugate,

$$\hat{h}'_i(\hat{\gamma}_i(\hat{\beta}_i(\hat{y}_i))) = \hat{\phi}_i(\hat{\gamma}_i)\hat{h}'_i(\hat{\beta}_i(\hat{y}_i)),$$

which is equivalent to

$$\begin{aligned} \hat{h}'_i(\hat{\alpha}_i(\hat{y}_i)) &= \hat{\phi}_i(\hat{\alpha}_i \cdot \hat{\beta}_i^{-1})\hat{h}'_i(\hat{\beta}_i(\hat{y}_i)) \\ &= \hat{\phi}_i(\hat{\alpha}_i) \cdot \hat{\phi}_i(\hat{\beta}_i^{-1})\hat{h}'_i(\hat{\beta}_i(\hat{y}_i)). \end{aligned}$$

We now prove (4.7.2). First, since \hat{f}_i is $2\epsilon_i$ -onto from $B_{R_0}(\hat{p}_i)$ to $B_{R_0+2\epsilon_i}(\hat{f}_i(\hat{p}_i))$ and \hat{f}_i is $\hat{\Gamma}_i$ -conjugate, \hat{f}_i is $2\epsilon_i$ -onto.

For any $R > R_0$ and any $\hat{x}_i, \hat{y}_i \in B_R(\hat{p}_i)$, we shall estimate $\left| |\hat{x}_i\hat{y}_i| - |\hat{f}_i(\hat{x}_i)\hat{f}_i(\hat{y}_i)| \right|$. Let $c : [0, l] \rightarrow \hat{M}_i$ ($l = |\hat{x}_i\hat{y}_i|$) be a minimal geodesic connecting \hat{x}_i and \hat{y}_i parametrized by arc length, and let $0 = t_0 < t_1 < \dots < t_s = l$ of $[0, l]$ be a partition such that $t_{j+1} - t_j = \frac{R_0}{2}$ ($0 \leq j < s-1$) and $t_s - t_{s-1} \leq \frac{R_0}{2}$. Then $s \leq \frac{2l}{R_0}$ and $|c(t_j)c(t_{j+1})| \leq \frac{R_0}{2}$. For each j , there is $\hat{\gamma}_j \in \hat{\Gamma}_i$ such that $B_{R_0}(\hat{\gamma}_j(\hat{p}_i))$ contains $c|_{[t_j, t_{j+1}]}$. Because \hat{f}_i is a $\hat{\Gamma}_i$ -conjugate and an $2\epsilon_i$ -GHA on $B_{R_0}(\hat{p}_i)$, we derive

$$\begin{aligned} & \left| |c(t_j)c(t_{j+1})| - |\hat{f}_i(c(t_j))\hat{f}_i(c(t_{j+1}))| \right| \\ &= \left| |\hat{\gamma}_j^{-1}(c(t_j))\hat{\gamma}_j^{-1}(c(t_{j+1}))| - |\hat{\phi}_i(\hat{\gamma}_j^{-1})\hat{f}_i(c(t_j))\hat{\phi}_i(\hat{\gamma}_j^{-1})\hat{f}_i(c(t_{j+1}))| \right| \\ &= \left| |\hat{\gamma}_j^{-1}(c(t_j))\hat{\gamma}_j^{-1}(c(t_{j+1}))| - |\hat{f}_i(\hat{\gamma}_j^{-1}c(t_j))\hat{f}_i(\hat{\gamma}_j^{-1}c(t_{j+1}))| \right| \\ &\leq 2\epsilon_i. \end{aligned}$$

Then

$$\left| \hat{f}_i(\hat{x}_i) \hat{f}_i(\hat{y}_i) \right| \leq \sum_j \left| \hat{f}_i(c(t_j)) \hat{f}_i(c(t_{j+1})) \right| \leq \left(1 + \frac{4}{R_0} \epsilon_i \right) |\hat{x}_i \hat{y}_i|.$$

To establish the opposite inequality, note that a minimal geodesic between $\hat{f}_i(\hat{x}_i)$ and $\hat{f}_i(\hat{y}_i)$ may not lie in the image of \hat{f}_i . Since \hat{f}_i is $2\epsilon_i$ -onto, we may replace the partition points by nearby points in $\hat{f}_i(\hat{M}_i)$. Similar to the above estimate we derive

$$|\hat{x}_i \hat{y}_i| \leq \left(1 + \frac{12}{R_0} \epsilon_i \right) \left| \hat{f}_i(\hat{x}_i) \hat{f}_i(\hat{y}_i) \right|.$$

Now (4.7.2) follows by taking $R_0 = 240d$. \square

Let $\pi : (\tilde{M}, \tilde{p}) \rightarrow (M, p)$ be the Riemannian covering space, and let $\Gamma = \pi_1(M, p)$. Observe that if $\text{diam}(M) \leq d$, then for any $R > 0$,

$$\frac{\text{vol}(B_R(\tilde{p}))}{\text{vol}(B_d(\tilde{p}))} \leq |\Gamma(R)| \leq \frac{\text{vol}(B_{R+d}(\tilde{p}))}{\text{vol}(B_d(p))},$$

and thus

$$h(M) = \lim_{R \rightarrow \infty} \frac{\ln \text{vol}(B_R(\tilde{p}))}{R} = \lim_{R \rightarrow \infty} \frac{\ln |\Gamma(R)|}{R}.$$

Proof of Theorem 4.6.

Let $\epsilon > 0$ satisfy that $\Gamma_i^\epsilon \xrightarrow{GH} G_0$ (see Lemma 1.13). By Theorem 2.5, $G_0 = e$. Then $\Gamma_i^\epsilon(\tilde{p}_i) \rightarrow \tilde{p}$, and thus Γ_i^ϵ is finite when i large. For $\gamma_i \in \Gamma_i(R)$, we may assume $\gamma_i \in \alpha_i \Gamma_i^\epsilon$. Observe that α_i can be chosen so that $\hat{\alpha}_i \in \hat{\Gamma}_i(R)$, where $\hat{\alpha}_i$ denotes the projection of α_i in $\hat{\Gamma}_i$. Assume that $|\Gamma_i^\epsilon| = C_i$. Then

$$(4.6.1) \quad |\hat{\Gamma}_i(R)| \leq |\Gamma_i(R)| \leq |\hat{\Gamma}_i(R)| \cdot |\Gamma_i^\epsilon| \leq C_i \cdot |\hat{\Gamma}_i(R)|.$$

We claim that

$$(4.6.2) \quad C_1 e^{(k-1)(1-\frac{\epsilon_i}{10d})R} \leq |\hat{\Gamma}_i(R)| \leq C_2 e^{(k-1)(1+\frac{\epsilon_i}{10d})R}$$

Combining (4.6.1) and (4.6.2), we derive

$$\left| \frac{1}{k-1} \cdot h(M_i) - 1 \right| = \left| \frac{1}{k-1} \cdot \lim_{R \rightarrow \infty} \frac{\ln |\Gamma_i(R)|}{R} - 1 \right| \leq \frac{\epsilon_i}{10d}.$$

We now verify (4.6.2). Let \hat{f}_i be in Lemma 4.7 and let $\hat{q}_i = \hat{f}_i(\hat{p}_i)$. By (4.7.2) for any $R > 0$,

$$\hat{G}(\hat{q}_i) \cap B_{(1-\frac{\epsilon_i}{10d})R}(\hat{q}_i) \subset \hat{f}_i \left(\hat{\Gamma}_i(R)(\hat{p}_i) \right) \subset \hat{G}(\hat{q}_i) \cap B_{(1+\frac{\epsilon_i}{10d})R}(\hat{q}_i).$$

Together with the fact that \hat{f}_i is $\hat{\Gamma}_i$ -conjugate, if the isotropy group $\hat{G}_{\hat{q}_i} = e$, then we get

$$(4.6.3) \quad |\hat{G}(\hat{q}_i) \cap B_{(1-\frac{\epsilon_i}{10d})R}(\hat{q}_i)| \leq |\hat{\Gamma}_i(R)| \leq |\hat{G}(\hat{q}_i) \cap B_{(1+\frac{\epsilon_i}{10d})R}(\hat{q}_i)|$$

Without loss of generality, we may assume that discrete \hat{G} acts freely on $B_{2\delta}(\hat{p})$ for some $\delta > 0$, and for i large, $\hat{q}_i = \hat{f}_i(\hat{p}_i)$ is in $B_\delta(\hat{p})$. By counting points in $\hat{G}(\hat{q}_i) \cap B_R(\tilde{p})$, we get

$$(4.6.4) \quad \frac{\text{vol}(\underline{B}_R^{-1})}{\text{vol}(\underline{B}_d^{-1})} \leq |\hat{G}(\hat{q}_i) \cap B_R(\hat{q}_i)| \leq \frac{\text{vol}(\underline{B}_R^{-1})}{\text{vol}(\underline{B}_\delta^{-1})}, \quad B_R(\hat{q}_i) = \underline{B}_R^{-1}.$$

By now, (4.6.2) follows from (4.6.3) and (4.6.4). \square

Proof of Theorem 0.5.

The proof is similar to the proof of Theorem 4.6, because $\dim(M) = n$. Hence, we will only briefly describe the proof.

First, since $\dim(M) = n$, $G_0 = e$, and since $\Gamma_i^\epsilon \xrightarrow{GH} e$, by Lemma 2.3 we conclude that for i large, $\Gamma_i^\epsilon = e$. By Lemma 1.13, we see that $\hat{\Gamma}_i = \Gamma_i / \Gamma_i^\epsilon \cong G / G_0 = G$. Assume $(h_i, \phi_i, \phi_i^{-1})$ be ϵ_i -equivariant GHG with $\epsilon_i \rightarrow 0$, where $\phi_i : \Gamma_i \rightarrow G$ is an isomorphism.

Following the proof of Lemma 4.7 with $\hat{M}_i = \tilde{M}_i$ and $\hat{X} = \tilde{X} = \tilde{M}$, via the center of mass method we construct a Γ_i -conjugate map, $\tilde{f}_i : (\tilde{M}_i, \tilde{p}_i, \Gamma_i) \rightarrow (\tilde{M}, \tilde{p}, G)$, such that (4.7.1) and (4.7.2) hold. By the estimate for $\hat{\Gamma}_i$ in the proof of Theorem 4.6, we get the desired result. \square

Proof of Corollary 0.6.

(0.6.1) \Rightarrow (0.6.3): By Theorem 0.5.

(0.6.3) \Rightarrow (0.6.2): By Theorem 4.1, \tilde{M} is close to \mathbb{H}^n . By Theorem 1.6 we see that (0.6.2) is satisfied.

(0.6.2) \Rightarrow (0.6.1): By Theorem B. \square

5. PROOF OF THEOREM 1.4

Our approach to Theorem 1.4 is based on the following functional criterion for warped product metric by Cheeger-Colding (see Theorem 5.1).

Let N be a Riemannian $(n-1)$ -manifold, let $k : (a, b) \rightarrow \mathbb{R}$ be a smooth positive function, and let $(a, b) \times_k N$ be the k -warped product whose Riemannian tensor is

$$g = dr^2 + k^2(r)g_N.$$

Then the function, $f = -\int_r^b k(u)du$, satisfies

$$\text{Hess}_f = k'(r)g.$$

Conversely, let (M, g) be a Riemannian manifold and let $r : M \rightarrow \mathbb{R}$ be the distance function to a compact subset of M . If there is a smooth function $f : M \rightarrow \mathbb{R}$ satisfying $\nabla f \neq 0$ and

$$\text{Hess}_f = h \cdot g$$

on $A_{a,b} = r^{-1}((a, b))$, where $h : M \rightarrow \mathbb{R}$ is a smooth function, then f is constant on each level set of r between a and b , and the Riemannian metric tensor in the annulus $A_{c,d}$ ($a < c < d < b$) is a warped product (cf. [CC1]),

$$g = dr^2 + (f'(r))^2 \tilde{g}.$$

Cheeger-Colding proved that if $\text{Hess}_f = k'(r)g$ holds approximately “in the L_1 -sense”, then the warped product structure of $A_{c,d}$ almost holds “in the Gromov-Hausdorff sense” [CC1].

Theorem 5.1 ([CC1]). *Let M be a Riemannian manifold with $\text{Ric}_M \geq -(n-1)H$, let r be a distance function to a compact subset in M , let $k : \mathbb{R} \rightarrow \mathbb{R}$ be a positive smooth function and let $f = -\int_r^b k(u)du$. For $0 < \alpha' < \alpha$, let $A_{a+\alpha, b-\alpha} \subset A_{a+\alpha', b-\alpha'}$ be two annuluses with respect to r . Let $d^{\alpha'}$ be the intrinsic metric in $A_{a+\alpha', b-\alpha'}$, and let $d^{\alpha', \alpha} = d^{\alpha'}|_{A_{a+\alpha, b-\alpha}}$. Assume*

(5.1.1) *for the metric $d^{\alpha', \alpha}$, $\text{diam}(A_{a+\alpha, b-\alpha}) \leq D$,*

(5.1.2) *for $0 < \delta < \alpha'$ and all $x \in r^{-1}(a + \alpha')$, there exists $y \in r^{-1}(b - \alpha')$ such that the intrinsic distance between x and y in $A_{a+\alpha'-\delta, b-\alpha'+\delta}$ satisfies*

$$d^{\alpha'-\delta}(x, y) \leq b - a - 2\alpha' + \delta.$$

(5.1.3) *there is $\tilde{f} : A_{a,b} \rightarrow \mathbb{R}$ satisfying*

$$(5.1.3.1) \quad |\tilde{f} - f| < \delta \text{ for all } x \in A_{a+\alpha', b-\alpha'},$$

$$(5.1.3.2) \quad \int_{A_{a,b}} |\nabla \tilde{f} - \nabla f| \leq \delta,$$

$$(5.1.3.3) \quad \int_{A_{a+\alpha', b-\alpha'}} |\text{Hess}_{\tilde{f}} - k'(r)g| \leq \delta,$$

Then there exists a metric space X , with $\text{diam}(X) \leq C(a, b, \alpha, \alpha', f, D, H)$, such that for the restricted metric $d^{\alpha, \alpha'}$ on $A_{a+\alpha, b-\alpha}$,

$$d_{GH}(A_{a+\alpha, b-\alpha}, (a + \alpha, b - \alpha) \times_k X) \leq \Psi(\delta | a, b, \alpha, \alpha', n, f, D, H).$$

We will only present a proof of Theorem 1.4 for $H < 0$, because a proof for $H = 0$ follows the same argument with a minor modification. By a rescaling, without loss of generality we assume $H = -1$.

From the proof of Theorem 5.1 (see Proposition 2.80 and Theorem 3.6 in [CC1]), we observe the following: If (5.1.2) holds on $B_\rho(q) \subset A_{a+\alpha, b-\alpha}(p)$, and one can find \tilde{f} such that (5.1.3) holds, then $d_{GH}(B_\rho(p), B_\rho(0, y)) < \Psi(\delta | \rho, n, f, H)$, where $B_\rho(0, y) \subset (a + \alpha, b - \alpha) \times_k X$ for some metric space X .

In view of Theorem 1.4, we choose $f = e^u, u(x) = |xp| - |pq|$, for some $q \in A_{L-R, L+R}(p)$ such that $B_\rho(q)$ satisfies (5.1.2), and \tilde{f} is the solution of

$$(5.2) \quad \begin{cases} \Delta \tilde{f} = ne^u, & \text{in } B_\rho(q); \\ \tilde{f} = f, & \text{on } \partial B_\rho(q). \end{cases}$$

Our strategy is to select balls in $A_{L-2R, L+2R}(p)$ such that (5.1.2) holds on each ball (see Lemmas 5.4 and 5.5), which also satisfies an additional property (see Lemma 5.8) so that we are able to verify (5.1.3) (see Lemma 5.9).

From the above discussion, the following theorem implies Theorem 1.4.

Theorem 5.3. *Let the assumptions be as in Theorem 1.4. Given $0 < \alpha < 1$, there are disjoint metric balls, $B_\rho(q_i) \subset A_{L-R, L+R}(p)$, satisfying (1.4.2) and the following:*

(5.3.1) *for $x \in B_\rho(q_i)$, there is $y \in \partial B_{L+R}(p)$ satisfying $|px| + |xy| \leq |py| + \Psi(\epsilon, L^{-1} | n, \rho, R)$;*

(5.3.2) *for each q_i , let $u(x) = |xp| - |q_i p|$, there is a smooth function \tilde{f} satisfying*

$$(5.3.2.1) \quad |\tilde{f} - e^u| < \Psi(\epsilon, L^{-1} | n, R, \rho) \text{ for all } x \in B_{(1-\alpha)\rho}(q_i).$$

$$(5.3.2.2) \quad \int_{B_\rho(q_i)} |\nabla \tilde{f} - \nabla e^u|^2 \leq \Psi(\epsilon, L^{-1} | n, R, \rho).$$

$$(5.3.2.3) \quad \int_{B_{(1-\alpha)\rho}(q_i)} |\text{Hess}_{\tilde{f}} - e^u|^2 \leq \Psi(\epsilon, L^{-1}|n, R, \alpha, \rho).$$

From now on, without mention explicitly we always assume the condition (1.5.1) and denote $\epsilon = \Psi(\epsilon|n, H, R)$.

Let E be a maximal subset of $\{q_i, B_\rho(q_i) \subset A_{L-R, L+R}(p)\}$ such that for all $q_{i_1} \neq q_{i_2} \in E$, $B_\rho(q_{i_1}) \cap B_\rho(q_{i_2}) = \emptyset$. Let $F = \bigcup_{q_i \in E} B_\rho(q_i)$. We shall choose $q_i \in E$ such that (5.3.1) and (5.3.2) hold on $B_\rho(q_i)$.

Lemma 5.4. *For L sufficiently large,*

$$\frac{\text{vol}(F)}{\text{vol}(A_{L-R, L+R}(p))} \geq (1 - \Psi(\epsilon, L^{-1}|n, \rho, R))e^{-(n-1)\rho} \cdot \frac{\text{vol}(\underline{B}_\rho^{-1})}{\text{vol}(\underline{B}_{2\rho}^{-1})}.$$

Proof. Let $G = \bigcup_{q_i \in E} B_{2\rho}(q_i)$. By the maximality of E , we have that,

$$A_{L-R+\rho, L+R-\rho}(p) \subset G.$$

For $L - R < r < L + R$, by (1.5.1) and the Bishop-Gromov relative volume comparison, we get

$$\frac{\text{vol}(\partial B_r(p))}{\text{vol}(\partial \underline{B}_r^{-1})} \geq (1 - \epsilon) \frac{\text{vol}(\partial B_{L-R}(p))}{\text{vol}(\partial \underline{B}_{L-R}^{-1})}.$$

Plugging the above into the integrant in the following quotient, together with the Bishop-Gromov relative volume comparison, we derive

$$\begin{aligned} (5.4.1) \quad \frac{\text{vol}(G)}{\text{vol}(A_{L-R, L+R}(p))} &\geq \frac{\text{vol}(A_{L-R+\rho, L+R-\rho}(p))}{\text{vol}(A_{L-R, L+R}(p))} \\ &= \frac{\int_{L-R+\rho}^{L+R-\rho} \text{vol}(\partial B_r(p)) dr}{\int_{L-R}^{L+R} \text{vol}(\partial B_r(p)) dr} \\ &\geq \frac{\int_{L-R+\rho}^{L+R-\rho} (1 - \epsilon) \frac{\text{vol}(\partial B_{L-R}(p))}{\text{vol}(\partial \underline{B}_{L-R}^{-1})} \text{vol}(\partial \underline{B}_r^{-1}) dr}{\int_{L-R}^{L+R} \frac{\text{vol}(\partial B_{L-R}(p))}{\text{vol}(\partial \underline{B}_{L-R}^{-1})} \text{vol}(\partial \underline{B}_r^{-1}) dr} \\ &= (1 - \epsilon) \frac{\text{vol}(\underline{A}_{L-R+\rho, L+R-\rho}^{-1})}{\text{vol}(\underline{A}_{L-R, L+R}^{-1})} \\ &\geq (1 - \Psi(\epsilon, L^{-1}|n, \rho, R))e^{-(n-1)\rho}. \end{aligned}$$

Again applying Bishop-Gromov relative volume comparison to the numerator of the quotient,

$$(5.4.2) \quad \frac{\text{vol}(F)}{\text{vol}(G)} \geq \frac{\sum_{q_i \in E} \text{vol}(B_\rho(q_i))}{\sum_{q_i \in E} \text{vol}(B_{2\rho}(q_i))} \geq \frac{\text{vol}(\underline{B}_\rho^{-1})}{\text{vol}(\underline{B}_{2\rho}^{-1})}.$$

The desired result follows from (5.4.1) and (5.4.2). \square

Next, we show that the balls in F on which (5.3.1) and (5.3.2) hold have a total volume almost equals to the volume of F .

Let $S \subset A_{L-R, L+R}(p)$ consist of interior points of minimal geodesics c_y from p to $y \in \partial B_{L+R}(p)$, i.e.,

$$S = \{x \in A_{L-R, L+R}(p) \cap c_y, y \in \partial B_{L+R}(p)\}.$$

Fixing $0 < \eta < 1$ (which will be specified later), let

$$E'(\eta) = \left\{ q_i \in E, \quad \frac{\text{vol}(B_\rho(q_i) \setminus S)}{\text{vol}(B_\rho(q_i))} < \eta \right\},$$

and let $F'(\eta) = \bigcup_{q_i \in E'(\eta)} B_\rho(q_i)$.

Lemma 5.5. *Let $F'(\eta)$ be defined in the above. Then*

$$\frac{\text{vol}(F'(\eta))}{\text{vol}(F)} \geq 1 - \eta^{-1} \Psi_1(\epsilon|n, R, \rho).$$

Proof. Since for any $q_i \in E \setminus E'(\eta)$,

$$\frac{\text{vol}(B_\rho(q_i) \setminus S)}{\text{vol}(B_\rho(q_i))} \geq \eta,$$

adding $\text{vol}(B_\rho(q_i))$ over q_i 's in $E \setminus E'(\eta)$ we derive

$$(5.5.1) \quad \frac{\text{vol}((F \setminus F'(\eta)) \setminus S)}{\text{vol}(F \setminus F'(\eta))} \geq \eta.$$

By Bishop-Gromov relative volume comparison and (1.5.1),

$$\begin{aligned} \frac{\text{vol}(S)}{\text{vol}(\underline{A}_{L-R, L+R}^{-1})} &\stackrel{(BG)}{\geq} \frac{\text{vol}(\partial B_{L+R}(p))}{\text{vol}(\partial \underline{B}_{L+R}^{-1})} \\ &\stackrel{(1.5.1)}{\geq} (1 - \epsilon) \frac{\text{vol}(\partial B_{L-R}(p))}{\text{vol}(\partial \underline{B}_{L-R}^{-1})} \\ &\stackrel{(BG)}{\geq} (1 - \epsilon) \frac{\text{vol}(A_{L-R, L+R}(p))}{\text{vol}(\underline{A}_{L-R, L+R}^{-1})} \end{aligned}$$

By (5.5.1) and Lemma 5.4,

$$\begin{aligned} \frac{\text{vol}(S)}{\text{vol}(A_{L-R, L+R}(p))} &= 1 - \frac{\text{vol}(A_{L-R, L+R}(p) \setminus S)}{\text{vol}(A_{L-R, L+R}(p))} \\ &\leq 1 - \frac{\text{vol}((F \setminus F'(\eta)) \setminus S)}{\text{vol}(F \setminus F'(\eta))} \frac{\text{vol}(F \setminus F'(\eta))}{\text{vol}(F)} \frac{\text{vol}(F)}{\text{vol}(A_{L-R, L+R}(p))} \\ &\leq 1 - \eta \cdot c(n, R, \rho) \cdot \frac{\text{vol}(F \setminus F'(\eta))}{\text{vol}(F)}. \end{aligned}$$

Combining the two estimates on $\text{vol}(S)$, we derive

$$\frac{\text{vol}(F'(\eta))}{\text{vol}(F)} \geq 1 - \eta^{-1} \cdot \epsilon \cdot c^{-1}(n, R, \rho).$$

□

Lemma 5.6. *Let the assumptions be as in Theorem 1.4, and let $r(x) = |px|$. Then*

$$\int_{A_{L-R, L+R}(p)} |\Delta r - (n-1)| \leq \Psi_2(\epsilon, L^{-1}|n, R).$$

Proof. Let the segment domain $M \setminus \text{Cut}(p)$ be equipped with the polar coordinates, let $\mathcal{A}(t, \theta) dt d\theta$ be the volume element. Since

$$\begin{aligned} \int_{A_{L-R, L+R}(p)} \Delta r &= \int_{L-R}^{L+R} \int_{S^{n-1}} \Delta r \mathcal{A}(t, \theta) d\theta dt \\ &= \int_{L-R}^{L+R} \int_{S^{n-1}} \frac{\mathcal{A}'(t, \theta)}{\mathcal{A}(t, \theta)} \mathcal{A}(t, \theta) d\theta dt \\ &= \int_{S^{n-1}} \int_{L-R}^{L+R} d\mathcal{A}(t, \theta) d\theta \\ &= \int_{S^{n-1}} (\mathcal{A}(L+R, \theta) - \mathcal{A}(L-R, \theta)) d\theta \\ &= \text{vol}(\partial B_{L+R}(p)) - \text{vol}(\partial B_{L-R}(p)), \end{aligned}$$

by (1.5.1) and

$$\lim_{L \rightarrow \infty} \frac{\text{vol}(\partial \underline{B}_{L-R}^{-1})}{\text{vol}(\underline{A}_{L-R, L+R}^{-1})} = \frac{n-1}{e^{2R(n-1)} - 1},$$

we derive

$$\begin{aligned} \int_{A_{L-R, L+R}(p)} \Delta r &= \frac{\text{vol}(\partial B_{L+R}(p)) - \text{vol}(\partial B_{L-R}(p))}{\text{vol}(A_{L-R, L+R}(p))} \\ &\geq \left((1-\epsilon) \frac{\text{vol}(\partial \underline{B}_{L+R}^{-1})}{\text{vol}(\partial \underline{B}_{L-R}^{-1})} - 1 \right) \frac{\text{vol}(\partial B_{L-R}(p))}{\text{vol}(A_{L-R, L+R}(p))} \\ &\geq \left((1-\epsilon) \frac{\text{vol}(\partial \underline{B}_{L+R}^{-1})}{\text{vol}(\partial \underline{B}_{L-R}^{-1})} - 1 \right) (1-\epsilon) \frac{\text{vol}(\partial \underline{B}_{L-R}^{-1})}{\text{vol}(\underline{A}_{L-R, L+R}^{-1})} \\ &\geq (1 - \Psi(\epsilon, L^{-1}|n, R))(n-1). \end{aligned}$$

Let $\underline{\Delta}$ denote the Laplacian on \mathbb{H}^n . By Laplace comparison, we derive

$$\begin{aligned} \int_{A_{L-R, L+R}(p)} \Delta r &\leq \int_{A_{L-R, L+R}(p)} \underline{\Delta} r \\ (5.6.1) \quad &= \int_{A_{L-R, L+R}(p)} (n-1) \frac{\cosh r}{\sinh r} \\ &\leq (1 + \Psi(L^{-1}|n, R))(n-1). \end{aligned}$$

The desired estimate then follows from the above two estimates for $\int_{A_{L-R, L+R}(p)} \Delta r$. \square

Lemma 5.7.

$$\int_F |\Delta r - (n-1)| \leq \Psi_3(\epsilon, L^{-1}|n, R, \rho).$$

Proof. By Lemma 5.4 and Lemma 5.6, we have that

$$\begin{aligned} \int_F \Delta r &= \frac{\text{vol}(A_{L-R, L+R}(p))}{\text{vol}(F)} \left(\int_{A_{L-R, L+R}(p)} \Delta r \right) - \frac{\int_{A_{L-R, L+R}(p) \setminus F} \Delta r}{\text{vol}(F)} \\ &\geq (1 - \Psi(\epsilon, L^{-1}|n, R))(n-1) \frac{\text{vol}(A_{L-R, L+R}(p))}{\text{vol}(F)} \\ &\quad - (n-1 + \Psi(L^{-1}|n, R)) \frac{\text{vol}(A_{L-R, L+R}(p) \setminus F)}{\text{vol}(F)} \\ &\geq (1 - \Psi(\epsilon, L^{-1}|n, R, \rho))(n-1). \end{aligned}$$

As in (5.6.1), we derive

$$\int_F \Delta r \leq (1 + \Psi(L^{-1}|n, R))(n-1).$$

□

Let $\Psi(\epsilon, L^{-1}|n, R, \rho) = \max \{ \Psi_1(\epsilon|n, R, \rho), \Psi_2(\epsilon, L^{-1}|n, R), \Psi_3(\epsilon, L^{-1}|n, R, \rho) \}$.

Lemma 5.8. *Let*

$$E''(\eta) = \left\{ q_i \in E, \int_{B_\rho(q_i)} |\Delta r - (n-1)| < \eta^{-1} \Psi(\epsilon, L^{-1}|n, R, \rho) \right\},$$

and let $F''(\eta) = \bigcup_{q_i \in E''(\eta)} B_\rho(q_i)$. Then

$$\frac{\text{vol}(F''(\eta))}{\text{vol}(F)} \geq 1 - \eta.$$

Proof. By Lemma 5.7, we derive

$$\begin{aligned} \Psi(\epsilon, L^{-1}|n, R, \rho) &\geq \int_F |\Delta r - (n-1)| \\ &= \frac{1}{\text{vol}(F)} \left(\sum_{E''(\eta)} \text{vol}(B_\rho(q_i)) \int_{B_\rho(q_i)} |\Delta r - (n-1)| \right. \\ &\quad \left. + \sum_{E \setminus E''(\eta)} \text{vol}(B_\rho(q_i)) \int_{B_\rho(q_i)} |\Delta r - (n-1)| \right) \\ &\geq \frac{1}{\text{vol}(F)} (0 + \eta^{-1} \Psi(\epsilon, L^{-1}|n, R, \rho) \text{vol}(F \setminus F''(\eta))) \\ &= \eta^{-1} \Psi(\epsilon, L^{-1}|n, R, \rho) \frac{\text{vol}(F \setminus F''(\eta))}{\text{vol}(F)}, \end{aligned}$$

i.e.,

$$\frac{\text{vol}(F \setminus F''(\eta))}{\text{vol}(F)} \leq \eta.$$

□

We now specify $\eta = \sqrt{\Psi(\epsilon, L^{-1}|n, R, \rho)}$. Then $F'(\eta) \cap F''(\eta)$ satisfies (1.4.2). By Bishop-Gromov relative volume comparison, (5.3.1) holds on balls in $F'(\eta)$.

To verify (5.3.2) on $B_\rho(q_i)$ for $q_i \in E'(\eta) \cap E''(\eta)$, we will use the standard comparison functions (see [Ch] for more details). Let

$$\underline{U}(r) = \int_0^r s n_H^{1-n}(s) \left(\int_0^s s n_H^{n-1}(u) du \right) ds,$$

$$\underline{G}(r) = \frac{1}{\omega^{n-1}} \int_r^\infty s n_H^{1-n}(s) ds,$$

where $\omega^{n-1} = \text{vol}(S_1^{n-1})$. For fixed $d > 0$,

$$\underline{U}_d(r) = \underline{U}(r) - \underline{U}(d), \quad \underline{G}_d(r) = \underline{G}(r) - \underline{G}(d),$$

$$\underline{L}_d(r) = -\frac{\underline{U}'(d)}{\underline{G}'(d)} \underline{G}_d(r) + \underline{U}_d(r).$$

Then $\underline{L}'_d(r) \leq 0$, $r \in [0, d]$, $\underline{\Delta} \underline{L}_d(r) = 1$, $\underline{\Delta} \underline{U}_d = 1$ and $\underline{U}'_d \geq 0$.

Lemma 5.9. (5.3.2) holds for each $q_i \in E''(\eta)$.

Proof. For $q = q_i \in E''(\eta)$, let $u(x) = |px| - |pq|$. By Lemma 5.8,

$$\int_{B_\rho(q)} |\Delta u - (n-1)| < \Psi(\epsilon, L^{-1}|n, R, \rho).$$

Let \tilde{f} be the solution of (5.2). Then,

$$\begin{aligned} \int_{B_\rho(q)} |\Delta(\tilde{f} - e^u)| &= \int_{B_\rho(q)} |ne^u - e^u(|\nabla u|^2 + \Delta u)| \\ &= \int_{B_\rho(q)} e^u |n - 1 - \Delta u| \\ &\leq \Psi(\epsilon, L^{-1}|n, R, \rho). \end{aligned}$$

By maximal principle, $\Delta(\tilde{f} - ne^{-2R}\underline{U}_{4R}(u + 2R)) \geq 0$, and $\Delta(\tilde{f} - ne^{2R}\underline{L}_{5R}(u + 2R)) \leq 0$, we have that $|\tilde{f} - e^u| \leq c(n, R, \rho)$. We then derive (5.3.2.2) as follows:

$$\begin{aligned} &\int_{B_\rho(q)} |\nabla \tilde{f} - \nabla e^u|^2 \\ &= \int_{B_\rho(q)} -\Delta(\tilde{f} - e^u)(\tilde{f} - e^u) \\ &\quad - \lim_{\delta \rightarrow 0} \frac{1}{\text{vol}(B_\rho(q))} \int_{\partial U_\delta \cap B_\rho(q)} \langle \nabla \tilde{f} - \nabla e^u, v \rangle (\tilde{f} - e^u) \\ &\leq \Psi(\epsilon, L^{-1}|n, R, \rho), \end{aligned}$$

where v is the normal vector to $\partial U_\delta \cap B_\rho(q)$, and ∂U_δ is a δ -tube neighborhood of the cut locus of p .

Let $h = |\nabla \tilde{f} - \nabla e^u|$, $\mathcal{F}_h(x, y) = \sup_\gamma \int_\gamma h$, where sup is taken over all minimal geodesics γ from x to y . Let $\Psi = \Psi(\epsilon, L^{-1}|n, R, \rho)$. For $x_1 \neq x_2 \in B_{(1-\Psi)\rho}(q)$, by Cheeger-Colding's segment inequality ([Ch], [CC1]),

$$\begin{aligned} \int_{B_{\frac{\Psi}{2}}(x_1) \times B_{\frac{\Psi}{2}}(x_2)} \mathcal{F}_h &\leq c(n, \rho) \left(\text{vol}(B_{\frac{\Psi}{2}}(x_1)) + \text{vol}(B_{\frac{\Psi}{2}}(x_2)) \right) \int_{B_\rho(q)} |\nabla \tilde{f} - \nabla e^u| \\ &\leq \Psi(\epsilon, L^{-1}|n, R, \rho). \end{aligned}$$

Then there exists $x'_1 \in B_{\frac{\Psi}{2}}(x_1)$, $x'_2 \in B_{\frac{\Psi}{2}}(x_2)$, such that $\int_{\gamma_{x'_1, x'_2}} h \leq \Psi(\epsilon, L^{-1}|n, R, \rho)$, i.e.,

$$\left| \left(\tilde{f}(x'_1) - e^{u(x'_1)} \right) - \left(\tilde{f}(x'_2) - e^{u(x'_2)} \right) \right| \leq \Psi(\epsilon, L^{-1}|n, R, \rho).$$

By Dirichlet-Poincaré inequality ([Ch]),

$$\int_{B_\rho(q)} |\tilde{f} - e^u| \leq c(n, R) \int_{B_\rho(q)} h \leq \Psi(\epsilon, L^{-1}|n, R, \rho).$$

Consequently we obtain (5.3.2.1).

Fixed $\alpha > 0$ small, by [CC1] we can choose a cut-off function ϕ satisfying

$$\begin{cases} \phi(x) = 1, & x \in B_{(1-\alpha)\rho}(q), \\ \phi(x) = 0, & x \in M \setminus B_{(1-\frac{\alpha}{2})\rho}(q), \end{cases} \quad |\nabla \phi|, |\Delta \phi| \leq c(n, \rho, \alpha).$$

By (5.3.2.1), (5.3.2.2) and Bochner's formula, we derive

$$\begin{aligned} \Psi(\epsilon, L^{-1}|n, R, \rho, \alpha) &\geq \frac{1}{2} \int_{B_\rho(q)} \Delta \phi (|\nabla \tilde{f}|^2 - \tilde{f}^2) \\ &= \frac{1}{2} \int_{B_\rho(q)} \phi \Delta (|\nabla \tilde{f}|^2 - \tilde{f}^2) \\ &= \int_{B_\rho(q)} \phi (|\text{Hess}_{\tilde{f}}|^2 + \text{Ric}(\nabla \tilde{f}, \nabla \tilde{f}) + \langle \nabla \Delta \tilde{f}, \nabla \tilde{f} \rangle \\ &\quad - \tilde{f} \Delta \tilde{f} - |\nabla \tilde{f}|^2) \\ &\geq \int_{B_\rho(q)} \phi (|\text{Hess}_{\tilde{f}} - e^u|^2 + ne^{2u} - (n-1)e^{2u} \\ &\quad + ne^{2u} - ne^{2u} - e^{2u}) - \Psi(\epsilon, L^{-1}|n, R, \rho, \alpha) \\ &\geq \int_{B_{(1-\alpha)\rho}(q)} |\text{Hess}_{\tilde{f}} - e^u|^2 - \Psi(\epsilon, L^{-1}|n, R, \rho, \alpha). \end{aligned}$$

□

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